

The Lorentz Triangle

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One of the pillars of relativity, the Lorentz factor appears in *length contraction*, in *time dilation* and in *mass increase*. It is also the primary factor determining *frames of reference* as depicted in the Lorentz transformations.

Perhaps the world's simplest definition¹ of the Lorentz factor is:

the ratio of the hypotenuse of a right-angled triangle to one of the other sides!

This paper will trace the origins of the Lorentz factor, it will show that, as derived by Lorentz, it is merely a specific case of a more general factor, and it will introduce a new tool, the Lorentz triangle, with which much of the confusion associated with this concept can be eliminated.

Lorentz was not the first to notice this factor. Maxwell, for example, is also said to have come across it in the context of his work. But its modern-day significance stems from Lorentz and his usage of it.

This can be traced back to the Michelson-Morley experiment (1887), and even to the first of Michelson's two initial experiments intended to verify the *ether*, conducted in 1881. In this experiment Michelson split a beam of light into two pencils of light, which he sent off at right angles to each other towards two reflectors at equal distances, then back to the source. One pathway, that in the direction of the earth's motion, was expected to require more time for the round trip than the other in the perpendicular direction.

No significant difference in the time taken was detected, however.

Lorentz proposed a correction to the initial mathematical calculations, which cast the previous result into doubt. Michelson therefore decided to repeat the experiment, allowing for this correction, but also improving the precision of his set-up in general.

Again, however, no significant difference in the time taken for the respective journeys of the pencils of light was detected.

¹ Or at least derivation.

And it was in response to this that Lorentz - and independently FitzGerald in Dublin - proposed the "solution" which initially concerns us here, *length contraction*.²

The following diagrams, as depicted by Michelson himself in his report on the famous Michelson-Morley experiment³, shows the essential difference in the perceived set-ups of the two experiments.

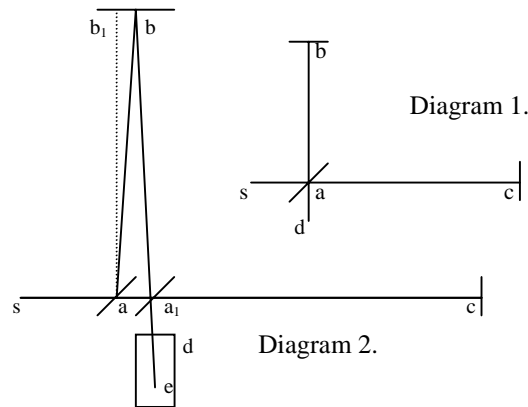


Fig. 1
 Illustrating the difference between
 the 1881 experiment (Diagram 1.) and
 the 1887 experiment (Diagram 2.)

Instead of assuming, as in 1881, that the perpendicular path travelled was aba (as shown in Diagram 1), it was now assumed in 1887 that the perpendicular path travelled was aba_1 (as shown in Diagram 2).

Since this is the difference that interests us in particular, we will now show it in a separate illustration, in which the triangle ABC in Fig. 2 (below) stands for ab_1b in Fig. 1, Diagram 2 (above):

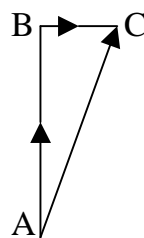


Fig. 2
 Outbound journey in the perpendicular direction

² Notice, incidentally, that unlike Michelson, who was prepared to accept the experimental results and their implications, Lorentz proposed *length contraction* as a means of compensating for Michelson's results and thereby retaining the *ether* theory.

³ *The Relative Motion of the Earth and the Luminiferous Ether*, Albert Abraham Michelson, American Journal of Science, 1881, 22: 120-129.

Whereas Michelson had initially assumed that the outbound path taken by the pencil of light in the perpendicular direction was AB (and also that the return journey was BA)⁴, it was now accepted that the outbound path was instead AC.

This adaptation is additionally depicted in the following illustration:

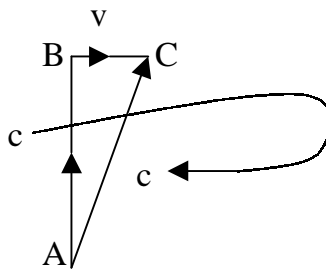


Fig. 3
Outbound journey in the perpendicular direction

The reason for the reassessment is also shown here, namely that the perpendicular path was, Lorentz believed, also required to accommodate a velocity component (v) in the direction of the earth's motion.⁵ Or put another way, the 1881 approach, for the perpendicular direction, was effectively based on $v=0$ in the direction of the earth's motion, whereas the 1887 approach was based instead on $v \neq 0$.

A third such illustration will now help us to complete our picture:

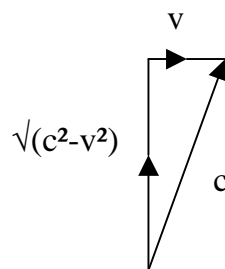


Fig. 4
Outbound journey in the perpendicular direction

Instead of the perpendicular direction having a value of c , as in Michelson's 1881 experiment, it now takes - by Pythagoras - the value of $\sqrt{c^2 - v^2}$.

⁴ Or ab_1 and b_1a in terms of Fig. 1, Diagram 2.

⁵ This alteration had to do with an argument in those days between the followers of Fresnel and Stokes, relating to the concept of aberration. An interesting point here, however, is that the alteration corresponds exactly to the change required in moving from the position that light takes on the velocity of its source, to the position that it does *not* take on the velocity of its source.

And this allows us to derive the Lorentz factor on the basis given right at the outset.

Before doing so, though, let us first *give* the Lorentz factor: $1 / \sqrt{(1-v^2/c^2)}$

This can also be written as: $(1-v^2/c^2)^{-1/2}$

In terms of Fig. 4, we derive the Lorentz factor by simply dividing the hypotenuse, c , by the value of the side AB, namely $\sqrt{(c^2-v^2)}$, as follows:

$$\begin{aligned}
 & c / \sqrt{(c^2-v^2)} \\
 = & c / \sqrt{c^2(1-v^2/c^2)} && \text{extracting } c^2 \text{ from the terms in brackets} \\
 = & c / c\sqrt{(1-v^2/c^2)} && \text{extracting } c, \text{ as the root of } c^2, \text{ from the remaining denominator} \\
 = & 1 / \sqrt{(1-v^2/c^2)} && \text{cancelling } c, \text{ above and below}
 \end{aligned}$$

And already we have it!

In the above illustration (Fig. 4) the terms involved have to do with velocity, namely the speed of light (c), the velocity of the earth (v) and the "truly" perpendicular equivalent ($\sqrt{(c^2-v^2)}$).

It is worth noticing, however, that we can actually derive the Lorentz factor from *any* such triangle, as will now be shown on the basis of the following illustrations:

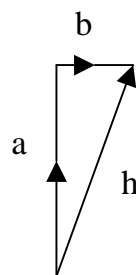


Fig. 5
The *Lorentz triangle*, as we shall call it
(h = hypotenuse, a & b are the shorter sides)

From this triangle we can redefine each of the sides in terms of the others:

$$\begin{aligned}
 a &= \sqrt{(h^2-b^2)} \\
 b &= \sqrt{(h^2-a^2)} \\
 h &= \sqrt{(a^2+b^2)}
 \end{aligned}$$

Substituting these values into our Lorentz triangle now gives us the following:

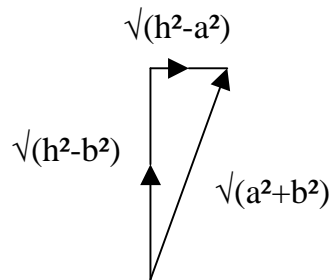


Fig. 6
The Lorentz triangle

For the *generalized* Lorentz factor, in terms of our Lorentz triangle, all we now have to do is to divide the hypotenuse by either of the other sides:

$$\begin{aligned}
 & \sqrt{a^2+b^2} / \sqrt{h^2-b^2} \\
 = & \sqrt{a^2+b^2} / \sqrt{h^2(1-b^2/h^2)} && \text{extracting } h^2 \text{ from the terms in brackets} \\
 = & \sqrt{a^2+b^2} / h\sqrt{(1-b^2/h^2)} && \text{extracting } h, \text{ as the root of } h^2, \text{ from the remaining denominator} \\
 = & h / h\sqrt{(1-b^2/h^2)} && \text{substituting } h \text{ for } \sqrt{a^2+b^2} \text{ in the numerator} \\
 = & 1 / \sqrt{(1-b^2/h^2)} && \text{cancelling } h, \text{ above and below}
 \end{aligned}$$

or

$$\begin{aligned}
 & \sqrt{a^2+b^2} / \sqrt{h^2-a^2} \\
 = & \sqrt{a^2+b^2} / \sqrt{h^2(1-a^2/h^2)} && \text{extracting } h^2 \text{ from the terms in brackets} \\
 = & \sqrt{a^2+b^2} / h\sqrt{(1-a^2/h^2)} && \text{extracting } h, \text{ as the root of } h^2, \text{ from the remaining denominator} \\
 = & h / h\sqrt{(1-a^2/h^2)} && \text{substituting } h \text{ for } \sqrt{a^2+b^2} \text{ in the numerator} \\
 = & 1 / \sqrt{(1-a^2/h^2)} && \text{cancelling } h, \text{ above and below}
 \end{aligned}$$

Generalizing still more, we can now state that the *true* Lorentz factor is:

$$1 / \sqrt{(1-s^2/h^2)}$$

where *s* is *either* of the shorter sides of a right-angled triangle and *h* is the hypotenuse.

Nor are we required to limit the terms used in our Lorentz triangle to velocities. Retaining the same triangle as used in our figures (2-6, above) we can instead substitute-in the distance terms applicable to Michelson's 1881 experiment, in which case we have the following illustration:

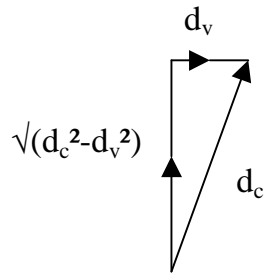


Fig. 7

Outbound distances in the perpendicular direction
 (d_c = the distance travelled by the light [in the given time],
 d_v = the distance travelled by the earth [in the given time],
 $\sqrt{d_c^2 - d_v^2}$ = the perpendicular equivalent)

Applying the Lorentz factor in terms of distance, we now have:

$$\begin{aligned}
 & d_c / \sqrt{d_c^2 - d_v^2} \\
 = & d_c / \sqrt{d_c^2 (1 - d_v^2 / d_c^2)} && \text{extracting } d_c^2 \text{ from the terms in brackets} \\
 = & d_c / d_c \sqrt{1 - d_v^2 / d_c^2} && \text{extracting } d_c, \text{ as the root of } d_c^2, \text{ from the remaining denominator} \\
 = & 1 / \sqrt{1 - d_v^2 / d_c^2} && \text{cancelling } d_c, \text{ above and below}
 \end{aligned}$$

One important thing no notice immediately is that the factor obtained, whether for velocity or for distance, is exactly the same, *when* the sides of the respective triangles are the same.

This realization allows us to return to velocity simply by adding in the *time* factor:

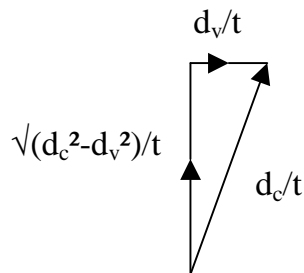


Fig. 8

Outbound velocities in the perpendicular direction
 (d_c/t = the distance travelled by the light in the given time,
 d_v/t = the distance travelled by the earth in the given time,
 $\sqrt{d_c^2 - d_v^2}/t$ = the perpendicular equivalent)

Notice that the time (t) here is the same for all three distances.

Now let's see, in terms of our Lorentz triangle, how Lorentz applies his factor.

Notice, first of all, that he is always applies the velocity factor, i.e. the Lorentz factor is always $1/\sqrt{1-v^2/c^2}$, never any of the other possibilities implied above.

This need not be a problem, however, since we have already seen that the ratios involved do not alter as long as the triangle does not alter, proportionately. As for associating the "velocity factor" with other terms, this is not a problem with time and distance, which are already closely bound up with it (velocity = distance travelled / time taken).

When Lorentz speaks of *length contraction*, however, the length being contracted is that of d_v in Fig. 7, above. This is equivalent to side b of Fig. 5. Now exactly why the ratio h/a (again Fig. 5) should be applied to reduce the length of side b is by no means mathematically obvious⁶, but we can at least now identify one aspect in this context: since $h/a > 1$, to reduce the length of side b this must be *divided* by the Lorentz factor.

Notice from Fig. 8, though, that this in turn results in a reduction of the velocity!

This *can* be compensated for if the time is also reduced by a similar factor. In this case, though, we would end up dividing both the numerator and the denominator by the Lorentz factor, which would be equivalent to not dividing at all!

As a taste of things to come, the situation with *mass increase* is one that is still less apparent ...

The Lorentz triangle is intended as a useful tool with which to approach an old topic - one that is often full of confusion - from a completely new angle. With it we can arguably see related consequences more clearly can ask, for example, questions like:

What is it about the proportion h/a that allows us to alter b on this basis?

or

Do *length contraction*, *time dilation* and *mass increase* all apply simultaneously?⁷

or

Do they apply individually?⁸

or

Do they apply at all?

⁶ The idea was initially introduced as a solution to the Michelson-Morley result, of course.

⁷ Which would mean having to half - or to third - the contribution originally attributed to *length contraction* alone, for each of the concepts now simultaneously involved.

⁸ And if so, when and why?