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Analytical Approach to Planar Two Body Problem and Three Laws of Kepler

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Abstract

This document is prepared on the purpose of explanation of how planetary motion and Kepler's Laws are derived with a simple approximation of 2-D planar analytical approach.

THREE LAWS OF KEPLER... HOW?

Consider the motion of a planet in the solar system which has a mass of m kilograms. Suppose at some reference time, $t = 0$ say, the planet is at a distance d away from the sun (which has a mass of M kilograms) and is moving with a certain (initial) velocity in a direction *not* along the line segment connecting the planet and the sun. to Describe the motion of the planet, we treat it as a single mass particle and introduce a polar coordinate system in the plane defined by the line segment d and the direction of the planet's motion at $t = 0$ with the sun at the origin (see Figure 1). The position of the planet relative to the sun at a later time t can be given by the position vector $\mathbf{r}(t) =< x(t), y(t) >$ directed from the origin to the planet at time t are $\mathbf{v}(t) \equiv \dot{\mathbf{r}}(t)$ and $\mathbf{a}(t) \equiv \ddot{\mathbf{r}}$, respectively to t so that the do operator is defined as$\dot{(\ )} \triangleq \frac{d(\ )}{dt}$ linear time derivative operator.

In this chapter we focus our attention on the main factor responsible for the planet's motion, namely, the gravitational attraction of the sun, and ignore secondary effects such as the influence of other planets in the solar system which are considerably smaller in mass than the sun. In *Newtonian Physics*, it is postulated that the acceleration of the planet is proportional to the force $\mathbf{F}(t)$ acting on the planet. More precisely, Newton's second law of motion for a constant mass particle stipulates $m\ddot{\mathbf{r}}(t) = \mathbf{F}$. Newton further postulated that the force acting on the planet is directly proportional to the product of the masses Mm, and inversely proportional to $r^2 = ||\mathbf{r}||^2$, where r is the distance between the sun and planet, and is pulling the planet toward the sun:

$$m\ddot{\mathbf{r}}(t) = -\frac{GMm}{r^2}\mathbf{a_r}(t)^{\dagger} \tag{1}$$

In equation 1 the quantity G is a constant of proportionality whose numerical value depends on the choice of units for *mass*, *length*, and *time* e.g. $G = 6.67 \times 10^{-8} \frac{cm^3}{g.sec^2}$. The postulate on $\mathbf{F}(t)$ is known as *Newton's Law of Universal Gravitation* or *The Inverse Square Law*.

The vector equation 1 is equivalent to two scalar equations for the distance function $\mathbf{r}(t)$ and the polar angle variable $\theta(t)$; these two quantities determine the position of the planet in the plane of the planet's motion at any instant time t. (That there is no out of plane motion is also a consequence of 1 and the initial condition for the omitted third axis $\mathbf{r}(t)\centerdot\mathbf{a_z}|_{t=0} = z(0) = 0^{\ddagger}$ and $\dot{\mathbf{r}}(t)\centerdot\mathbf{a_z}|_{t=0} = v_z(0) = 0$). With $\mathbf{r}(t) = r\mathbf{a_r} = r\cos\theta\mathbf{a_x} + rsin\theta\mathbf{a_y}$, we are going to deduce from equation 1 the following two scalar second order differential equations:

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†The vector $\mathbf{a_r}\,(\theta(t))$ defines exactly two-dimensional radial unit vector i.e. $\mathbf{a_r}\,(\theta(t)) \triangleq \mathbf{a_x}\cos\theta(t) + \mathbf{a_y}\sin\theta(t)$, where $\mathbf{a_x}$ and $\mathbf{a_y}$ are orthonormal unit vector set (basis) that can span all the 2-D Euclidian space and they are $constant$ over the 2-D Euclidian plane i.e. $\frac{\partial(\mathbf{a_{x,y}}(x,y))}{\partial(x)} = 0$ and $\frac{\partial(\mathbf{a_{x,y}}(x,y))}{\partial(y)} = 0$, $\forall(x,y) \in \mathbb{R}^2$

‡The $\centerdot$ operator is defined over $(\ )\centerdot(\ )$ $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Where n denotes dimension. It defines n dimensional Euclidian space inner product i.e. projection product exactly.

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$$
\begin{aligned}
2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 && (2)\\
\ddot{r} - r\dot{\theta}^2 &= \frac{-MG}{r^2} && (3)\\
& && (4)
\end{aligned}
$$

Now considering the figure 1 we are going to deduce the aforementioned equations of motion by using Newton's principle of motion i.e. $F(t) = ma(t)$

$$
\begin{aligned}
\mathbf{F}(t) &= m\mathbf{a}(t) && (5)\\
-\mathbf{a_r}\frac{GMm}{r^2} &= m\ddot{\mathbf{r}} && (6)\\
\mathbf{r(t)} &= ||\mathbf{r(t)}||\mathbf{a_r} && (7)\\
r(\ddot{t})\mathbf{a_r} &= \frac{d^2}{dt^2}\left[r(t)\mathbf{a_r}(\theta(t))\right]^{\S} && (8)\\
\frac{d^2}{dt^2}\left[r(t)\mathbf{a_r}(\theta(t))\right] &= \frac{d}{dt}\left[r\mathbf{a}_\theta\dot{\theta} + \dot{r}\mathbf{a_r}\right] = \dot{r}\dot{\theta}\mathbf{a}_\theta + r\ddot{\theta}\mathbf{a}_\theta - r\dot{\theta}^2\mathbf{a_r} + \ddot{r}\mathbf{a_r} + \dot{r}\dot{\theta}\mathbf{a}_\theta && (9)\\
-\frac{GM}{r^2} &= -r\dot{\theta}^2 + \ddot{r} && (10)\\
0 &= 2\dot{r}\dot{\theta} + r\ddot{\theta} && (11)
\end{aligned}
$$

During the derivation of the aforesaid equations of 10 and 11, the position vector is differentiated w.r.t. time t from equations 8 to 9 and the derivative is substituted in the equation 6.

Given the (initial) position and velocity vectors of the planet at some reference $t = 0$,

$$
\begin{aligned}
\mathbf{r}(0) &= \mathbf{r_o} = r_o\mathbf{a_r}(\mathbf{0}) = r_o\mathbf{a_{r_o}} && (12)\\
\dot{\mathbf{r}}(0) &= \mathbf{v_o} = v_o\mathbf{a_r}(\mathbf{0}) + v_o\mathbf{a}_\theta(\mathbf{0}) = v_o\mathbf{a_{r_o}} + v_o\mathbf{a}_{\theta_\mathbf{o}} && (13)
\end{aligned}
$$

with $\mathbf{a_{ro}} = \cos\theta_\mathbf{o}\mathbf{a_x} + \sin\theta_\mathbf{o}\mathbf{a_y}$ and $\mathbf{a}_{\theta\mathbf{o}} = -\sin\theta_\mathbf{o}\mathbf{a_x} + \cos\theta_\mathbf{o}\mathbf{a_y}$, the two coupled nonlinear ODE 2 and 3 determine $r(t)$ and $\theta(t)$ for all later time i.e. $t > 0$, instead of obtaining explicit expressions for r and θ as functions of t, we will deduce from the two differential equations of motion, 2 and 3, the three empirical laws describing the motion of the planets proposed by *J. Kepler* in early 17^{th} century after laboring for 22 years over available observational data. In order to appreciate *Kepler's Laws*, we must keep in mind that along the planet's orbit, the radial position of the planet is completely determined by its angular position. Loosely speaking, we may solve $\theta = H(t)$ for t in terms of θ, i.e. $t = T(\theta)$, and use it to eliminate t from $r = R(t)$ to get $r = R(T(\theta)) = r(\theta)$.

I. Kepler's Second Law

Since r is no identically zero $\forall t > 0$, we can multiply 2 by r to get

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = (\dot{r^2})\dot{\theta} + r^2\ddot{\theta} = \frac{d\left(r^2\dot{\theta}\right)}{dt} = 0$$

and therewith

$$\int_{t'=0}^{t'=t}\frac{d\left(r^2\acute{\theta}^{\P}\right)}{dt'}dt' = r^2\dot{\theta} = \frac{\rho_o}{m} \qquad (14)$$

§By considering the polar coordinate basis vectors of $\mathbf{a_r} \triangleq \cos\theta\mathbf{a_x} + \sin\theta\mathbf{a_y}$ time derivative of $\mathbf{a_r}$ can be written as $\dot{\mathbf{a_r}} = \frac{d}{dt}\left(\mathbf{a_r}\left[\theta(t)\right]\right) = \frac{d}{dt}\left(\cos\theta\mathbf{a_x} + \sin\theta\mathbf{a_y}\right) = \frac{d}{dt}\left(\cos\theta\mathbf{a_x}\right) + \frac{d}{dt}\left(\sin\theta\mathbf{a_y}\right) = \left(-\sin\theta\mathbf{a_x} + \cos\theta\mathbf{a_y}\right)\frac{d\theta}{dt} \equiv \mathbf{a}_\theta\dot{\theta}$, it can be easily seen that $\dot{\mathbf{a_r}} = \mathbf{a}_\theta\dot{\theta}$ and $\dot{\mathbf{a}_\theta} = -\mathbf{a_r}\dot{\theta}$.

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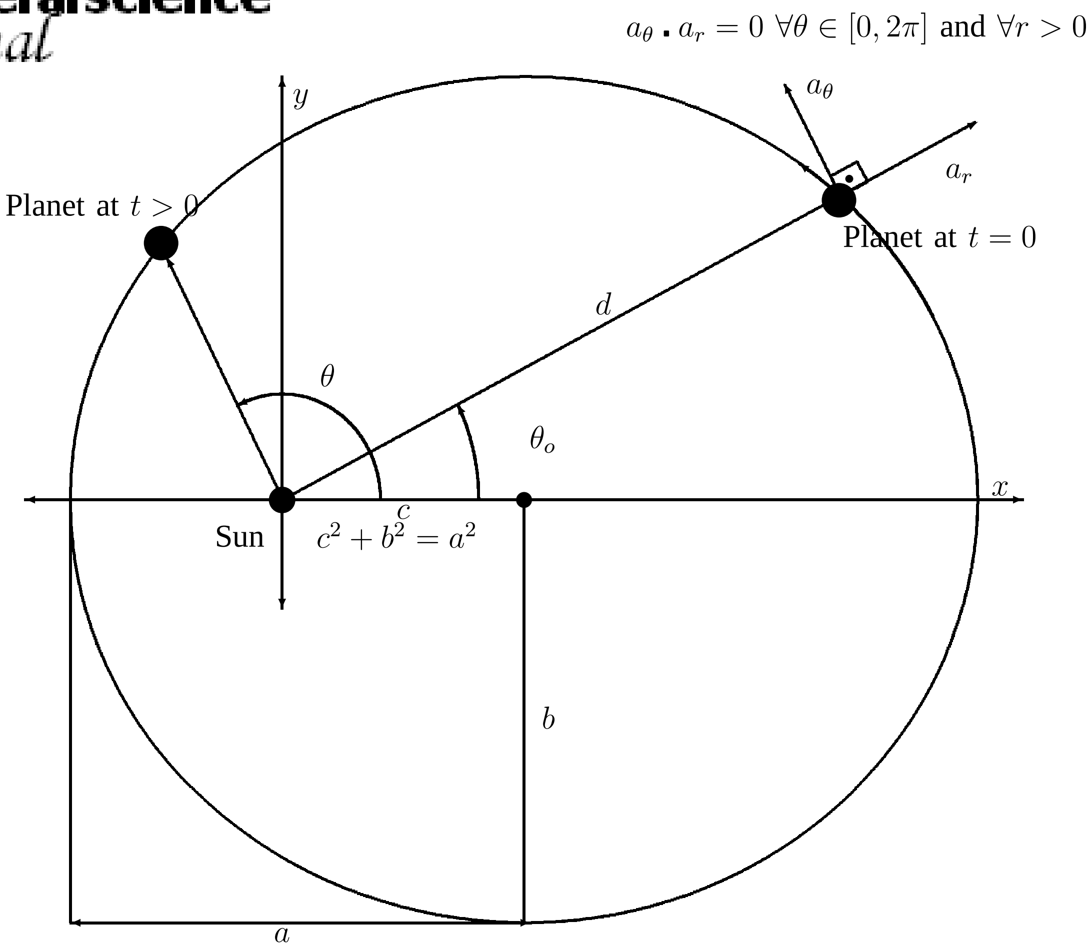

Fig. 1. A 2-D illustration of the planetary motion. At the marked focus of the ellipse sun exist with a mass of M.

where ρ_o is a constant of integration to be determined by the initial position and velocity of the planet at $t = 0$. (In fact, we have $\frac{\rho_o}{m} = r_o v_{\theta_o}$). The quantity $mr^2\dot{\theta}$ is known as *angular momentum* of the planet; equation 14 tells us that the angular momentum of the planet is the same $\forall(t > 0)$, or *angular momentum is conserved*.

Suppose that at time $t = t_1$, the planet is at point $P_1 = (x(t_1), y(t_1))$ corresponding to $r = r_1$ and $\theta = \theta_1$, and it gets to a point P_2 corresponding to $r = r_2$ and $\theta = \theta_2$ at a later time for $t_2 > t_1$. Recall that r may be thought of as a function of θ for points on the orbit. Then the position vector $\mathbf{r}$ sweeps out a sector during the time interval $\Delta t \equiv t_2 - t_1$ with an area,

$$\Delta A = [Area]_{t=t_1}^{t=t_2} = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=0}^{r=r(\theta)} r dr d\theta = \frac{1}{2} \int_{\theta=\theta_1}^{\theta=\theta_2} r^2 d\theta = \int_{t=t_1}^{t=t_2} r^2 \dot{\theta} dt \tag{15}$$

where $r(\theta)$ is the plane curve traced out by the planet's orbit. By equation 14, the integrand is independent of t and we get

$$\Delta A = \frac{\rho_o}{2m} \Delta t \tag{16}$$

Equation 16 is a mathematical statement of *Kepler's second law*:

$$\begin{array}{l} The\ position\ vector\ of\ the\ planet's\ orbit \\ sweeps\ out\ sections\ of\ equal\ areas\ in\ equal \\ time\ intervals. \end{array} \tag{17}$$

¶ Here $\acute{()}$ denotes the time derivative according to replaced and primed dummy variable of time i.e. $\acute{()} \triangleq \frac{d(\ )}{dt'}$.In order to carry out running integration $\forall t$ in the interval, parameter swap is performed within the range of $t' \in (0, t)$ $dt = dt'$

The formula 16 actually tells us a little more about this area than Kepler's original statement of the law; it gives a specific relation between this area and angular momentum per unit mass of the particular planet.

II. Kepler's First and Third Laws

We now use 14 to eliminate the time dependent derivative $\dot{\theta}$ from 3to get

$$\ddot{r} - \frac{\rho_o^2}{m^2}\frac{1}{r^3} = \frac{GM}{r^2} \tag{18}$$

Equation 18 is a second order nonlinear ODE in which the independent variable t does not appear explicitly. The usual method of solution for this class of ODE is nothing more that a straight forward analytical approach to *second order in-homogenous linear ordinary differential equation*. In the normal case once we have $r(t)$, we can get $\dot{\theta}(t)$ from 14 and then $\theta(t)$ by a simple integration. For the shape of the planet's orbit however, it suffices to have r as a function of θ. We can ge $r(\theta)$ directly from 14 and 18 without first solving 18 for r(t). This is accomplished by observing that

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = -\frac{\rho_o}{mr^2}\frac{dr}{d\theta} = -\frac{\rho_o}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \tag{19}$$

where we have used 14 to eliminate $\dot{\theta}$, and correspondingly

$$\ddot{r} = -\frac{\rho_o}{m}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\dot{\theta} = -\frac{\rho_o^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \tag{20}$$

so we that we can write 18 as

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) = \frac{GMm^2}{\rho_o^2} \tag{21}$$

The solution of 21 is immediate since it is a linear second order ODE with coefficients for $\frac{1}{r}$:

$$\frac{1}{r} = A\cos\theta + B\sin\theta + \frac{GMm^2}{\rho_o^2} \tag{22}$$

The constants of integration in equation 22 , namely, A and B are then determined by the initial conditions $r = r_o$ and $\frac{dr}{d\theta} = \frac{r_o v_{r_o}}{v_{\theta_o}}$ at $\theta = \theta_o$ (see equation 12 and 13). For the case $v_{r_o} = 0$, the resulting solution may be written as

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos\theta - \theta_o} \tag{23}$$

where $\frac{rho_o^2}{GMm^2} = a(1-e^2)$ and $\frac{\rho_o^2}{r_o GMm^2} = 1 + e$ For $0 < e < 1$, the formula 23 is he mathematical counterpart of the *Kepler's first law*:

$$The\ orbit\ of\ the\ planet\ around\ the\ the\ sun\ is\ an\ ellipse. \tag{24}$$

In expression 23, the quantity e is the *eccentricity* of the ellipse, the smaller e, the closer the ellipse is to a circle; the quantity a is the *semi-major axis* of the ellipse (see Figure).

Let T be the time it takes for the planet to complete one revolution of its orbit. We know from 16 that

$$\frac{\rho_o T}{2m} = Ellipse\ Area = \pi a^2\sqrt{1-e^2} \tag{25}$$

But $\sqrt{1-e^2} = \frac{\frac{\rho_o}{m}}{\sqrt{GMa}}$; therefore, we have;

$$T^2 = \frac{4\pi^2 a^3}{GM} \tag{26}$$

$$\begin{array}{l} The\ square\ of\ the\ orbital\ period\ is\ proportional \\ to\ the\ third\ power\ of\ the\ semi-major\ axis \\ of\ the\ elliptical\ orbit. \end{array} \tag{27}$$

Just as in the case of the second law, equatins 23 and 26 contain much more information than Kepler's original statement of first and third law. Moreover, Newton's results are qualitative and precise; they can be (and have been) used to make prediction on measurable quantities such as position and velocity which characterize the motion of the planet.

III. An Applied Mathematical Perspective

Kepler, empiricist, spent nearly his entire professional life analyzing the data amassed by *Tacho Brahe* on the motion of the planets and managed to infer from these data three rules which govern the behavior of the planets as they orbit around the sun. His accomplishment is epochal in the development of astronomy, a discipline which tries to understand the universe in which we live. But by comparison, Newton's results on the same problem as given by 16, 26, and 23 and others not presented here (e.g. expressions for r and θ in terms of time etc.) derived from his laws of motion and universal gravitation are much quantitative, mote precise and more substantial. There is a strong correlation between high precision and depth of the conclusions on the one hand and their mathematical deduction from the fundamental principles and laws of nature on the other hand. The fact that Newton is a major figure in mathematics (as well as in physics) was no accident; he had to develop the mathematical tools needed to achieve the breadth and depth of his work in physics. To strive for quantitative precision, modern curricula in science and engineering require their students to be exposed to more and more mathematics. Even such relatively extensive exposure is often not enough to cope with many new problems which require new mathematical methods of clever use of available mathematical techniques; it often someone who is well schooled in applied mathematics beyond the materials prescribed in science and engineering curricula to do either.

Quantitative precision aside, there is another fundamental difference between the results of Kepler and Newton. Kepler's undeniably outstanding contribution consists of factual statements beautifully synthesized from observational data. Newton's remarkable results are meant to help us to understand there facts; why is the orbit elliptical? why the equal areas rule exist? and why the three half power law for the orbital period? Newton told us why: because the dynamics of all mass particles, be it a planet or an electron, are governed by his three laws of motion and his law of universal gravitation! Newton's laws are fundamental to mechanics of mass particles, rigid bodies, and deformable media, not just planetary motion alone. Yet, it was Kepler's synthesis which provided Newton with a stimulus for his construct, namely, his law of universal gravitation.

References

[1] Frederic Y.M. Wan, *Lecture Notes for Applied Mathematics for Continuous Systems: Some Mathematical Models and Their Analysis*. The University of British Columbia, Vacouver, Canada. May, 1985