# AN EXPLICIT FORMULA FOR $\pi(x)$ IN THE FORM OF A SUM OVER THE NON-TRIVIAL ZEROS OF RIEMANN ZETA FUNCTION

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## Abstract:

In this paper we discuss a method to express the Prime counting function as a "sum" over Non-trivial zeros of Riemann Zeta function, using techniques from Analytic Number Theory, also we apply our results to the sum over primes of any function $\sum_{p \le r} f(p) = \int_2^r dt f(t) d\pi$

- *Keywords:* = Riemann zeta function zeros, sum over primes, Logarithmic integral.

## Expression for $\pi(x)$ As a sum over zeros:

If we use the expressions of the Prime Counting function via Mellin transform:

$$P(s) = s\int_0^\infty dx \pi(x) x^{-s-1} \quad \to \pi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \frac{P(s)}{s} \quad (1)$$

Where P(s) is the "Prime zeta function" (sum over the inverse powers of primes) that satisfies:

$$\ln\zeta(s) = \sum_{n=1}^{\infty} \frac{P(ns)}{n} \qquad \to \quad P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln\zeta(sn) \quad (2) \quad , \quad P(s) = \sum_p p^{-s}$$

Where we have introduced the Mertens and Möbius functions related by the sum: $M(x)=\sum_{n\le x}\mu(n)$

Using the Abel sum-formula with the Möbius function:

$\sum_{n=1}^{\infty}\mu(n)f(n)=\int_{-\infty}^{\infty}dxM'(x)f(x)$ and the identity (contour integration)

$$M_0(x)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}ds\frac{x^s}{s\zeta(s)}=\sum_{\rho}\frac{x^{\rho}}{\zeta'(\rho)\rho}-2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!\zeta(2n+1)n}\left(\frac{2\pi}{x}\right)^{2n}\begin{cases}M(x)-\frac{1}{2}\mu(x) & \text{iff } x\in Z^{+}\\ M(x) & \text{Otherwise}\end{cases}$$

(3)

And $M_0'(x)=M'(x)=\frac{dM}{dx}\quad x\neq Z^{+}$

In case that there are no multiple Non-trivial roots of the Riemann Zeta function

$|\zeta'(\rho)|\neq 0$ , $\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^n n^{-s}$ $\Re\text{e}(s)>0$ , $\zeta(1)=\infty$

If we combine expressions (1) and (2) involving P(s) and perform the Mellin transform, Using the properties of Laplace transform with t=ln(x) and the fact that:

$$\frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\frac{\ln\zeta(ns)}{ns}x^s=Li(x^{1/n})-\sum_{\rho}Li(x^{\rho/n})-\ln 2+\int_{x^{1/n}}^{\infty}dt\frac{1}{t(t^2-1)\ln t}$$

For n=1 we obtain the “Riemann Prime counting function”

$$J(x)=\sum_{n=1}^{\infty}\frac{\pi(x^{1/n})}{n}=Li(x)-\sum_{\rho}Li(x^{\rho})-\ln 2+\int_{x}^{\infty}dt\frac{1}{t(t^2-1)\ln t}\qquad(4)$$

$Li(x)=\int_{2}^{x}\frac{dt}{\ln t}$ (Offset Logarithmic integral)

Then using (1) (2) and (3) the Prime counting function reads:

$$\pi(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\int_{1}^{\infty}ds\ln\zeta(ns)\frac{M_0'(n)}{ns}$$

(Mellin inverse transform plus Abel sum formula)

$$\pi(x)=\int_1^\infty \frac{du}{u^2}G(u)\left(Li(x^{1/u})-\sum_\rho Li(x^{\rho/u})-\ln 2+\int_{x^{1/u}}^\infty dt\frac{1}{t(t^2-1)\ln t}\right)$$

$$\sum_\rho \frac{u^\rho}{\zeta'(\rho)}+\sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)!\zeta(2n+1)n}\left(\frac{2\pi}{u}\right)^{2n}=G(u)$$

And, for the sum over primes of a certain function f(x) using Abel formula and the result above, we have that:

$$\sum_{p\le r} f(p)=\int_2^r dt\pi'(t)f(t)d\pi=\int_2^r dx\int_1^\infty \frac{du}{u^2}G(u)\left(\frac{x^{(\frac{1}{u}-1)}}{\ln(x)}-\sum_\rho \frac{x^{\frac{\rho}{u}-1}}{\ln(x)}-\frac{1}{x(x^{\frac{2}{u}}-1)\ln(x)}\right)f(x)$$

A faster method to get the Prime counting function is using the definitions for J(x) and $M_0$ (x), involving the Möbius inversion formula and Abel sum formula is:

$$J(x)=\sum_{n=1}^\infty \frac{\pi(x^{1/n})}{n} \qquad \to \qquad \pi(x)=\sum_{n=1}^\infty \frac{\mu(n)}{n}J(x^{1/n})$$

With (4) and $\sum_{n=1}^\infty \mu(n)f(n)=\int_{-\infty}^\infty dxM'(x)f(x)$ you could recover $\pi(x)$.

The sum over the Non-trivial roots of $\zeta(s)$, since it is only conditionally convergent, must be summed in order of increasing $\Im m[\rho]$:

$$\sum_\rho f(x^\rho)=\sum_{\Im m[\rho]>0}[f(x^\rho)+f(x^{1-\rho})]$$

The sum over the Non-trivial zeros , (those different from $\zeta(-2n)=0$ n=1,2,3,4,5,.....) has a deep connection with operator theory, if Riemann Hypothesis is correct then $\rho_n=\frac{1}{2}+iE_n$ Where the $E_n$ are the Eigenvalues of a certain Hermitian operator T , if we could also consider the prime numbers to be some kind of "Eigenvalues" of a certain operator P then we could make the connection:

$$J(x)=\sum_{n=1}^\infty \frac{\pi(x^{1/n})}{n}=Tr\left[\hat{1}Li(x)-Li(\sqrt{x}x^{i\hat{T}})-\hat{1}\ln 2+\hat{1}\int_x^\infty dt\frac{1}{t(t^2-1)\ln t}\right] \text{ and}$$

$$Tr\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \frac{e^{sx}}{s} e^{-s\hat{P}}\right] = \pi(x) \qquad \hat{1} = \text{Identity operator}\ ,\ \hat{T}^{\dagger} = \hat{T}$$

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