

AN APPROACH TO $\pi(x)$ AND OTHER ARITHMETICAL FUNCTIONS BY VARIATIONAL PRINCIPLES

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ABSTRACT: In this paper we present a method to get the prime counting function $\pi(x)$ and other arithmetical functions than can be generated by a Dirichlet series, first we use the general variational method to derive the solution for a Fredholm Integral equation of first kind with symmetric Kernel $K(x,y)=K(y,x)$, after that we find another integral equations with Kernels $K(s,t)=K(t,s)$ for the Prime counting function and other arithmetical functions generated by Dirichlet series, then we could find a

solution for $\pi(x)$ and $\sum_{n=1}^x a(n) = A(x)$, solving $\delta J[\phi] = 0$ for a given functional J, so the problem of finding a formula for the density of primes on the interval $[2,x]$, or the calculation of the coefficients for a given arithmetical function $a(n)$, can be viewed as some "Optimization" problems that can be attacked by either iterative or Numerical methods (as an example we introduce Rayleigh-Ritz and Newton methods with a brief description)

Also we have introduced some conjectures about the asymptotic behavior of the series

$\Xi_n(x) = \sum_{p \leq x} p^n = S_n(x)$ for $n > 0$, and a new expression for the Prime counting function in terms of the

Non-trivial zeros of Riemann Zeta and its connection to Riemman Hypothesis and operator theory.

- *Keywords:* =PNT (prime number theorem), Variational Calculus, Maxima and minima, Integral Transforms.

1. VARIATIONAL METHODS IN NUMBER THEORY:

It was Euler, in the problem of “Brachistochrone” (shortest time) or curve of fastest descent, who introduced the preliminaries of what later would be called, “Calculus of Variations”, he solved the problem minimizing the integral below, where “t” is the time employed by the particle to go from (0,0) to another point on the plane (x, y) $x \neq 0$ below (0,0), using Newtonian mechanics he found the expression:

$$\int_a^b dt \sqrt{\frac{1+(\dot{y})^2}{y}} = t \quad (\text{for constant gravity } g=9.8 \text{ m/s}^2 \text{ and ignoring friction}) \quad (1.1)$$

Then he got that the minimum of the integral above was a differential equation describing a cycloid, later Lagrange used this Calculus of Variations to describe the mechanics of a particle introducing the Lagrangian and the action functional S, whose extremum gave precisely the equations of motion for the particles:

$$\frac{\delta S}{\delta q} = 0 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \quad S = \int_a^b dt L(q_j, \dot{q}_j, t) \quad \frac{dq_j}{dt} = \dot{q}_j \quad (1.2)$$

The equations above are the Euler-Lagrange equations of motion for the system defined by the Lagrangian L, this formulation is equivalent to Newton second law, although its use is more extended, as you only need to know the Kinetic part of the particle (usually of the form $T = \frac{1}{2} \sum_{i,j}^N Q^{ij} \dot{q}_i \dot{q}_j$) and a potential V related to the force $\vec{F} = -\vec{\nabla}(V)$ is the force of the system, Here Q is an Hermitian Matrix.

Now we can ask ourselves if we can generate a similar Variational principle for number theory, for the case of a Fredholm equation of second kind, with $K(x,y)=K(y,x)$ we have:

$$J[f] = a \int_a^b dx f^2(x) + \int_a^b g(x) f(x) dx - \int_a^b dx \int_a^b dy K(x, y) f(x) f(y) \quad (1.2)$$

Taking the functional derivative.

$$\frac{\delta J[f]}{\delta f(x)} = 0 = af(x) + g(x) - \int_a^b dy K(x, y) f(y) \quad (1.3)$$

$$a \in \square \quad \frac{\delta F}{\delta \phi(x)} = \lim_{\varepsilon \rightarrow 0} \frac{F[\phi + \varepsilon \delta(x-y)] - F[\phi]}{\varepsilon} \quad (1.4)$$

So we have derived for this especial case the Euler-Lagrange equation for this Integral equation, for the cases of the prime counting function and an arithmetical function that can be generated via Dirichlet series of the form:

$$g(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \quad A(x) = \sum_{x \geq n > 1} a(n) \quad (1.5)$$

We can give the 2 Functional, so their maximum or minimum are precisely some integral equations defining these arithmetical functions: (a=0)

$$J_0[E] = \int_0^{\infty} \frac{\ln \zeta(2s)}{2s} E(s) dx - \int_0^{\infty} ds \int_0^{\infty} dt (e^{2st} - 1)^{-1} E(s)E(t) \quad E(t) = E_0 \pi(e^t) \quad (1.6)$$

$$J_1[\phi] = \int_0^{\infty} \frac{g(s+c)}{s+c} \phi(s) dx - \int_0^{\infty} ds \int_0^{\infty} dt e^{-st} \phi(s)\phi(t) \quad \phi(t) = A_0 A(e^t) e^{-ct} \quad (1.7)$$

Minimizing these Functional $\delta J_{0,1} = 0$ with respect to E(s) and $\phi(s)$ and making the change of variable $t=\ln(x)$ we get the usual integral equation for $\pi(x)$ and A(x) namely:

$$g(s+c) = (s+c) \int_0^{\infty} dx \frac{A(x)}{x^{s+1+c}} \quad \frac{\ln \zeta(2s)}{2s} = \int_2^{\infty} dx \frac{\pi(x)}{x(x^{2s}-1)} \quad , \quad \prod_p (1-p^{-2s}) = \frac{1}{\zeta(2s)} \quad (1.8)$$

With $c>0$ and $s>1/2$ (the election of $g(s+c)$ and $2s$ inside the second integral is to avoid the singularities inside the integrals due to a pole of the Riemann zeta function at $s=1$), this allows us to study the prime counting function and other arithmetical functions by using the Optimization techniques, including some iterative methods (gradient-descent, Newton method...) to calculate their “shape”, as an example of these iterative methods we can get the Maximum or Minimum for a given Functional in the form:

$$\phi_{n+1} = \phi_n - \frac{\delta J[\phi_n]}{\delta^2 J[\phi_n]} \quad \| \phi_{n+1} - \phi_n \| < \varepsilon \quad n \rightarrow \infty \quad (1) \quad J=J_0, J_1 \quad (1.9)$$

So we “stop” the process when the condition on the right (1.9) is fulfilled, of course we need an initial “guess” function, for the exponential prime counting function with J_0 we

can use the exponential integral $E(t) = E_i(x) = -\int_{-x}^{\infty} dt \frac{e^{-t}}{t}$ or simply $E(t) = \exp(t)/t$, using

some upper and lower bounds for $\pi(x)$, for our arithmetical functions if their sum satisfies that $A(x) = O(v(x))$, for a known function $v(x)$ we could use our trial function in the form, $v(e^t)e^{-ct}$ and take into account that $A(n) - A(n-1) = a(n)$

The use of an appropriate trial function to begin with the iterative process is not casual, as the Newton iterative method could fail and diverge for certain trial functions, other inconveniences are that to compute Newton method we should need to know the value of the first and second functional derivative, which is not always available or easy to obtain, other optimization methods although slower in convergence are more secure, generally we will use a Gradient method for infinite-dimensional space to get an initial and convergent set of trial functions and then, to improve the convergence, we will use a Newton-like method.

For the case that our arithmetical function is generated by a power series:

$$G(x) = \sum_{n=0}^{\infty} x^n p(n) \quad |x| < R \text{ (Radius of convergence)} \quad \text{Where } B(t) = \sum_{n \leq t} p(n)$$

We can use the functional J_1 with $\phi(t)=B(t)$ with $c=0$ and $G(e^{-s})=g(s)$, to get the values of $p(n)$ and $B(t)$, by obtaining the maximum or minimum of this J .

Another useful method is if we choose our trial function to be of the form

$$\phi(t) \approx \sum_{n=0}^N C_n W_n(t) \text{ Where the functions } W_n(t) \text{ are known, so our Functional } J \text{ becomes}$$

$$J_{0,1} = J(C_0, C_1, C_2, \dots, C_N) \text{ And to obtain the } C_n \text{ we need to solve the system}$$

$$\frac{\partial J}{\partial C_n} = 0 \quad n=0,1,2,3,\dots,N \quad \text{With } N \rightarrow \infty \text{ (Rayleigh-Ritz method)}$$

For the study of Prime counting function $\pi(e^t)$ when $t \rightarrow \infty$, we could choose

$$W_n(t) = E_i\left(\frac{t}{n}\right), \quad n > 0 \text{ so we can introduce some correction to the PNT}$$

$\pi(e^t) \rightarrow E_i(t) = Li(e^t)$ for big t , or simply if we use the trial function $aE_i(bt)$ and take

$$\frac{\partial J}{\partial a} = 0 = \frac{\partial J}{\partial b}, \text{ we calculate the Hessian symmetric Matrix with elements:}$$

$$\frac{\partial^2 J}{\partial C_i \partial C_j} = H_{i,j} \text{ We will have a Maximum or a Minimum depending on if } H \text{ is negative or positive definite.}$$

2 .ON THE ASYMPTOTIC BEHAVIOR OF THE SUMS OVER PRIMES, BEYOND THE PRIME NUMBER THEOREM:

Now that we have given a method to obtain $\pi(x)$ by Variational methods, after that we could study every sum over primes:

$$\pi'(x) = \frac{d\pi(x)}{dx} = \sum_p \delta(x-p) \quad \sum_{p \leq x} f(p) = \int_2^x dt \pi'(t) f(t) \quad p = \text{sum over primes.} \quad (2.1)$$

Where we have introduced the Dirac delta function with definition and properties:

$$\int_a^b dx f(x) \delta(x-a) = f(a) \quad 2\pi \delta(x-a) = \int_{-\infty}^{\infty} d\omega e^{\pm i\omega(x-a)} \quad (2.2)$$

Now we would be interested in some asymptotic behavior for the cases $f(t)=t^n$ as a conjecture, we have that the probability of a random integer number being prime is $1/\ln(x)$ then we have the approximate (asymptotic) relationships:

$$\Xi_n(x) = \sum_{p \leq x} p^n \rightarrow \sum_{k=2}^x \frac{k^n}{\ln(k)} \rightarrow \int_2^x \frac{t^n dt}{\ln(t)} = Li(x^{n+1}) \quad n=0, n>0 \quad (2.3)$$

$$Li(x^{n+1}) = \ln(\ln(x)) + \sum_{k=1}^{\infty} (n+1)^k \frac{\ln^k(x)}{m.m!} + \gamma \quad (2.4)$$

The last identity is found on tables for indefinite integrals, with γ the Euler-Mascheroni constant, for $n=0$ we get the Prime Number Theorem (PNT) $\pi(x) \rightarrow x/\ln(x)$, for $n=1$ we get the asymptotic relation:

$$\Sigma(n) = \sum_{i=1}^n p_i = \Xi_1(p_n) \quad \text{Setting } p_n \rightarrow n \ln(n) \quad (3) \quad Li(x^2) \rightarrow \frac{x^2}{2 \ln(x)} \quad Li(x) = \int_2^x \frac{dt}{\ln(t)}$$

(3) is the asymptotic expression for the n -th prime, we use the European convention so $Li(2)=0$, expanding the $Li(x^2)$, keeping only the first term and using the prime number theorem to give an expression for n -th prime, we get : $\Sigma(n) \rightarrow \frac{n^2 \ln(n)}{2}$

For the case $n=-1$ the terms inside the sum over k cancel, so we get that the Harmonic prime series (as shown before, by Euler and others) diverges in the form $\ln(\ln(x))$ as $x \rightarrow \infty$, Or if our conjecture is valid, we can study the growth-rate of the series :

$$\Xi_n(x) = \sum_{p \leq x} p^n \rightarrow \pi(x^{n+1}) \rightarrow Li(x^{n+1}) \quad \sum_{p \leq x} p^n = \int_2^x dt \pi'(t) t^n \rightarrow \pi(x^{n+1}) \quad (2.5)$$

$$\text{Differentiating both sides: } \frac{\pi'(x)}{\pi'(x^{n+1})} \rightarrow (n+1) \quad \text{With } \pi'(x) = \frac{d\pi}{dx} \rightarrow \frac{1}{\ln(x)} \quad (\text{PNT})$$

The last is how the derivative of the prime counting function behaves for big x , using the relation (2.4) with an function $F(x)$ that is analytic near $x=0$ and has the limit $\lim_{x \rightarrow \infty} F(x) = \infty$, then the asymptotic expression for the sum :

$$\sum_{p \leq x} F(p) \rightarrow \sum_{n=0}^{\infty} c(n) Li(x^{n+1}) \quad c(n) = \frac{d^n F}{dx^n}(0) \frac{1}{n!} \quad (2.6)$$

Where we have used the asymptotic notation $f(x) \rightarrow g(x)$ meaning that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$

For other convergent prime sums, first we use the identity:

$$\sum_{n=0}^{\infty} (-1)^n [\pi(n) - \pi(n-1) + 1] f(n) = 2f(2) - \sum_p f(x) + \sum_{n=0}^{\infty} (-1)^n f(n) \quad (2.7)$$

To obtain a relation between a prime series and an alternating series, valid when both, the alternating series and the sum over all primes for $f(x)$ converge, the main purpose of

this is that we can accelerate the convergence of the prime series using only a few values of $\pi(n)$ $n < 100$, as an example putting $f(x) = x^{-s}$.

$$\sum_{n=0}^{\infty} (-1)^n [\pi(n) - \pi(n-1) + 1] n^{-s} = 2^{1-s} - \sum_p p^{-s} + \eta(s) \quad (2.8)$$

Where we have introduced the Prime Zeta function $P(s) = \sum_p p^{-s}$

$\eta(s) = (1 - 2^{1-s})\zeta(s)$ Dirichlet Zeta function, The Euler (forward) transform for alternating series is given by the expression:

$$\sum_{n=0}^{\infty} (-1)^n f(n) \approx \sum_{n=0}^{\infty} (-1)^n \frac{\Delta^n f(0)}{2^{n+1}} \quad \Delta f(n) = f(n+1) - f(n) \quad (2.9)$$

Euler method allows you to accelerate the convergence of an alternating series like (5) with general terms $(-1)^n [\pi(n) - \pi(n-1) + 1] f(n)$ and $(-1)^n f(n)$ providing that $f(n) > f(n+1)$, if Euler transform is not good or converges worse than the initial series, we could use the backward Euler transform with the Backward difference operator $\nabla f(n) = f(n) - f(n-1)$, the main purpose of using Euler transform is to be able to

compute the expression $2f(2) - \sum_p f(x) + \sum_{n=0}^{\infty} (-1)^n f(n)$. For a given f with a small

error, knowing only a few values of the Prime counting function, so we can extract the behavior or approximate the sum for the series over all the primes, calculating the sum of the alternating series with general term $(-1)^n f(n)$, which is in general, easier to calculate, this expression to evaluate prime sums can also be applied to products.

Even if the sum is divergent, we could use the functional equation for Riemann function relating $\zeta(s) \rightarrow \zeta(1-s)$ so using the definition of the Prime zeta function in terms of the Möbius formula and Riemann Zeta :

$$\sum_p p^s (-1)^r \ln^r(p) = \sum_{n=1}^{\infty} \mu(n) \frac{d^{r-1}}{ds^{r-1}} \left(\frac{\zeta'(-ks)}{\zeta(-ks)} \right) \quad ks \neq 2, 4, 6, \dots \quad s > 0 \quad (2.10)$$

and r being a positive integer bigger than 1, we can consider the infinite sum on the right being the regularized value of the divergent prime sum on the left

From equation (1.8) we can find a Lambert series representation for the prime counting function, introducing the notation $E(n) = \pi(e^n)$, and $s = -\ln x$ we can find the Lambert series for Prime number counting function:

$$\sum_{n=1}^{\infty} (E(n) - E(n-1)) n \frac{x^n}{1-x^n} = -x \frac{\partial}{\partial x} \left(\frac{\zeta(-\ln x)}{1} \right) = \frac{\zeta'(-\ln x)}{\zeta(-\ln x)} \quad (2.11)$$

3. A PROPOSED APPROXIMATE METHOD TO CALCULATE $\pi(X)$ VIA A CONTOUR INTEGRAL AND Θ - OPERATOR .

A method derived from Euler to calculate the prime number counting function (not in a strict or rigorous sense) using integral transforms and the “Theta” operator can be achieved, first we define:

$$e^{(a-1)\Theta} f(x) = f(ax) \quad \Theta(x) = x \frac{d}{dx} \quad (3.1)$$

The operator relation above is easy to shown, if we expand the function $f(ax)$ at the point $a=1$ we get (supposing that $f(ax)$ is infinitely differentiable in the variable a)

$$f(ax) = f(x) + \sum_{n=1}^{\infty} \frac{(a-1)^n}{n!} \frac{d^n f}{dx^n} = e^{(a-1)\Theta} f(x) \quad (3.2)$$

Using the “Riemann prime counting function” expression:

$$J(x) = \sum_{n=1}^{\infty} \frac{\pi(x^{1/n})}{n} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} ds \frac{\ln \zeta(s)}{s} x^s \quad (3.3)$$

Setting, $x=\exp(t)$ and applying the Laplace transforms and its properties

$$F(s) = L[\pi(e^t)] = \int_0^{\infty} dt \pi(e^t) e^{-st} \quad (3.4)$$

$$\frac{\ln \zeta(s)}{s} = \sum_{n=1}^{\infty} F(ns) = \sum_{n=1}^{\infty} e^{(n-1)\Theta(s)} [F(s)] = \frac{1}{1-e^{\Theta(s)}} [F(s)] \quad \lim_{n \rightarrow \infty} F(ns) = 0 \quad (3.5)$$

Now applying the operator function $(1-e^{\Theta(s)}) = -\sum_{n=1}^{\infty} \frac{\Theta^n}{n!}$ to both sides and inverting the Laplace transform, knowing that $t= \ln(x)$, we finally get for the prime counting function:

$$\pi(x) \approx \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} ds Z(s) x^s \quad \text{With} \quad Z(s) = (1-e^{\Theta^{-1}(s)}) \frac{\ln \zeta(s)}{s} \quad (3.6)$$

$$\text{Here} \quad \Theta^{-1} f(x) \rightarrow \int \frac{dx}{x} f(x) \quad (3.7)$$

Although we have used the Θ operator, Euler worked in a similar formula to obtain a finite result for the sum $1+2+3+4+5+6+7+8+\dots \rightarrow -1/12$, he began with the expression

$$f(x) + f(x+1) + f(x+2) + \dots = S \quad (3,8)$$

And used the identity $e^{aD} f(x) = f(x+a)$ Which can be proved by expanding $f(x+a)$ into a Taylor series near the point $a=0$,

$$D = \frac{d}{dx}, \text{ So for the sum (3.8), applying D and 1/D then } \frac{D}{1-e^D} F(x) = -\sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{d^n F(0)}{dx^n}$$

where $\frac{dF}{dx} = f(x)$ putting $f(x)=x$ we would get the result $-\frac{B_2}{2} = -\frac{1}{12} = S$ (3.9)

Similar results can be obtained for $f(x)=x^n$ with n a positive integer, including the relation between the negative values of Riemann zeta function and the Bernoulli numbers $|B_n| = n |\zeta(1-n)|$ (3.10)

Using Lie-Group theory is easy to check that D and Θ are the generators for the one dimensional translations $x' = x + c$ and dilations $x' = cx$

4. EXPRESSION FOR $\pi(x)$ AS A SUM OVER ZEROS:

If we use the expressions of the Prime Counting function via Mellin transform:

$$P(s) = s \int_0^{\infty} dx \pi(x) x^{-s-1} \rightarrow \pi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{P(s)}{s} \quad (4.1)$$

Where $P(s)$ is the “Prime zeta function” (sum over the inverse powers of primes) that satisfies:

$$\ln \zeta(s) = \sum_{n=1}^{\infty} \frac{P(ns)}{n} \rightarrow P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln \zeta(sn) \quad (4.2)$$

Where we have introduced the Mertens and Möbius functions related by the sum:

$M(x) = \sum_{n \leq x} \mu(n)$. Now using the Abel sum-formula with the Möbius function:

$$\sum_{n=1}^{\infty} \mu(n) f(n) = \int_{-\infty}^{\infty} dx M'(x) f(x) \quad \text{and} \quad M'_0(x) = M'(x) = \frac{dM}{dx} \quad (4.3)$$

$$M_0(x) = \sum_{\rho} \frac{x^{\rho}}{\zeta'(\rho)\rho} - 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!\zeta(2n+1)n} \left(\frac{2\pi}{x}\right)^{2n} \begin{cases} M(x) - \frac{1}{2}\mu(x) & \text{iff } x \in Z^+ \\ M(x) & \text{Otherwise} \end{cases} \quad (4.4)$$

In case that there are no multiple Non-trivial roots of the Riemann Zeta function so

$$|\zeta'(\rho)| \neq 0, \quad \zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^n n^{-s} \quad \Re(s) > 0, \quad \zeta(1) = \infty \quad (4.4)$$

If we combine expressions (4.1-2) involving $P(s)$ and perform the Mellin transform, using the next identity obtained by Riemann:

$$\frac{n}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\ln \zeta(ns)}{ns} x^s = Li(x^{1/n}) - \sum_{\rho} Li(x^{\rho/n}) - \ln 2 + \int_{x^{1/n}}^{\infty} dt \frac{1}{t(t^2-1)\ln t} \quad (4.5)$$

For $n=1$ we obtain the ‘‘Riemann Prime counting function’’

$$J(x) = \sum_{n=1}^{\infty} \frac{\pi(x^{1/n})}{n} = Li(x) - \sum_{\rho} Li(x^{\rho}) - \ln 2 + \int_x^{\infty} dt \frac{1}{t(t^2-1)\ln t} \quad (4.6)$$

Then the Prime counting function in terms of the non-trivial zeroes would read:

$$\pi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \int_1^{\infty} ds \ln \zeta(ns) \frac{M_0'(n)}{ns} \quad (4.7)$$

(Mellin inverse transform plus Abel sum formula)

$$\pi(x) = \int_1^{\infty} \frac{du}{u^2} G(u) \left(Li(x^{1/u}) - \sum_{\rho} Li(x^{\rho/u}) - \ln 2 + \int_{x^{1/u}}^{\infty} dt \frac{1}{t(t^2-1)\ln t} \right) \quad (4.8)$$

$$\sum_{\rho} \frac{u^{\rho}}{\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)! \zeta(2n+1)n} \left(\frac{2\pi}{u} \right)^{2n} = G(u) \quad (4.9)$$

And, for the sum over primes of a certain function $f(x)$ using Abel formula and the result above, we have that:

$$\sum_{p \leq r} f(p) = \int_2^r dt \pi'(t) f(t) d\pi = \int_2^r dx \int_1^{\infty} \frac{du}{u^2} G(u) \left(\frac{x^{(\frac{1}{u}-1)}}{\ln(x)} - \sum_{\rho} \frac{x^{\frac{\rho-1}{u}}}{\ln(x)} - \frac{1}{x(x^{\frac{2}{u}}-1)\ln(x)} \right) f(x) \quad (4.10)$$

A faster method to get the Prime counting function is using the definitions for $J(x)$ and $M_0(x)$, and then using the Möbius inversion formula together with Abel sum formula :

$$J(x) = \sum_{n=1}^{\infty} \frac{\pi(x^{1/n})}{n} \quad \rightarrow \quad \pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n}) = \int_1^{\infty} \frac{dM_0(u)}{u} J(x^{1/u}) \quad (4.11)$$

The sum over the Non-trivial roots of $\zeta(s)$, must be summed in order of increasing $\Im m[\rho]$ imaginary part to avoid some problems with convergence:

$$\sum_{\rho} f(x^{\rho}) = \sum_{\substack{\rho \\ \Im m[\rho] > 0}} [f(x^{\rho}) + f(x^{1-\rho})] \quad (4.12)$$

The sum over the Non-trivial zeros , (those different from $\zeta(-2n) = 0 \quad n=1,2,3,4,5,\dots$) has a deep connection with operator theory, if Riemann Hypothesis is correct then $\rho_n = \frac{1}{2} + iE_n$ Where the E_n are the Eigenvalues of a certain Hermitian operator T , if we could also consider the prime numbers to be some kind of ‘‘Eigenvalues’’ of a certain operator P then we could make the connection:

$$J(x) = \sum_{n=1}^{\infty} \frac{\pi(x^{1/n})}{n} = Tr \left[\hat{1}Li(x) - Li(\sqrt{x}x^{\hat{T}}) - \hat{1} \ln 2 + \hat{1} \int_x^{\infty} dt \frac{1}{t(t^2 - 1) \ln t} \right] \quad (4.13)$$

$$Tr \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{e^{sx}}{s} e^{-s\hat{P}} \right] = \pi(x) \quad \hat{1} = \text{Identity operator} , \hat{T}^{\dagger} = \hat{T} \quad (4.14)$$

Another curious relation involving Riemann Hypothesis, Logarithmic integral and Möbius function comes from combining Riemann-Siegel formula:

$$\frac{\pi(x) - Li(x)}{\frac{\sqrt{x}}{\ln x}} \approx -1 - 2 \sum_{\gamma} \gamma^{-1} \sin(\gamma \ln x) \quad (4.15)$$

Where the sum on the right is over the imaginary part of Non-trivial zeros of the form $\frac{1}{2} + i\gamma \quad \gamma \in C$, in case RH is true then all the imaginary parts are real, hence if we combine formula (4.15) and formulae (4.5-11) then we find an expression for the Logarithmic integral in terms of a sum over the imaginary parts of the zeros:

$$\frac{\int_1^{\infty} \frac{du}{u^2} G(u) \left(Li(x^{1/u}) - \sum_{\rho} Li(x^{\rho/u}) \right) - Li(x)}{\sqrt{x} / \ln x} \approx -1 - 2 \sum_{\gamma} \frac{\sin(\gamma \ln x)}{\gamma} \quad (4.16)$$

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