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ORTHOGONAL POLYNOMIALS, MOMENT PROBLEM AND THE RIEMANN XI-FUNCTION $\xi(1/2+iz)$

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ABSTRACT: In this paper we study a set of orthogonal Polynomials $\{p_n(x)\}$ with respect a certain given measure $\omega(x) = d\alpha(x)$ related to the Taylor series expansion of the Xi-function $\frac{\xi(1/2+iz)}{\xi(1/2)}$, this paper is based on a previous conjecture by Carlon and Gaston related to the fact that Riemann Hypothesis (with simple zeros) is equivalent to the limit $\lim_{n\to\infty}\frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2+iz)}{\xi(1/2)}$ for a certain set of orthogonal Polynomials, we study the 'Hamburger moment problem' $\frac{a_{2n}}{a_0}(2n)! = \mu_{2n} = \int_{-\infty}^{\infty} d\alpha(x)x^{2n}$ for even 'n' and 0 for n odd here the moments $\{\mu_n\}$ are related to the power series expansion of Xi-function $\frac{\xi(1/2+iz)}{\xi(1/2)} = \sum_{n=0}^{\infty}(-1)^n\frac{a_{2n}}{a_0}z^{2n}$, we also give the integral representation for the generating function $\sum_{n=0}^{\infty}(-1)^n\mu_{2n}z^{2n} = f(z)$, in terms of the Laplace transform of $\frac{\xi(1/2+iz)}{\xi(1/2)}$, and in the end of the paper we study the connection of our orthogonal polynomial set $\{p_n(x)\}$ with the Kernel $K(x,y) = C\frac{\Xi(x)\Xi'(y)-\Xi(y)\Xi'(x)}{x-y}$, $\Xi(z) = \xi(1/2+iz)$

through all the paper we will use the simplified notation $\{p_n(x)\} = \{p_n(x)\}_{n=0}^{\infty}$,$\{a_n\} = \{a_n\}_{n=0}^{\infty}$

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ORTHOGONAL POLYNOMIALS

A set of Polynomials $\{p_n(x)\}$ are orthogonal with respect to a certain measure on $L^2(a,b)$ if $\int_a^b d\alpha(x)p_n(x)p_m(x) = h_n\delta_{m,n}$ ($\delta_{m,n}$ Kronecker delta) , in this paper we will investigate a certain set of Polynomials $\{p_n(x)\}$ on $(-\infty,\infty)$ respect to a certain even measure $\omega(x) = d\alpha(x) = \omega(-x)$, we have used mainly the notation of Szego's [8] book in order to describe the properties of these orthogonal Polynomials.

We will study a set of orthogonal Polynomials with respect to a certain measure with the following properties

- The measure $\omega(x) = d\alpha(x)$ is positive on the interval $(-\infty,\infty)$, $\int_{-\infty}^{\infty} d\alpha(x) > 0$ with $d\alpha(x) \ge 0$,also it satisfies the functional equation $\omega(x) = \omega(-x)$.
- The orthogonal Polynomials (Szego page 29) according to the property above will be odd or even depending on the value of argument 'n' $p_{2n+1}(x) = -p_{2n+1}(-x)$ and for the even case $p_{2n}(x) = p_{2n}(-x)$ this fact is related to the definition of the moment problem $\mu_{2k+1} = 0$ for odd 'k' and $\mu_{2k} > 0$ for k even
- All the roots of $p_n(x)$ are REAL and distincts on the interval $(-\infty,\infty)$
- All the orthogonal polynomials $\{p_n(x)\}$ follow a 3-point recursive formula
  $p_n(x) = (a_n x + b_n)p_{n-1}(x) - c_n p_{n-2}(x) \quad n = 2,3,4,5,.....$
  For some constants a_n and b_n , this allows a faster computation for every $p_n(x)$.
- Since the measure is even, the 'Hamburger moment problem' associated to this measure , will have only non-zero even moment $\int_{-\infty}^{\infty} d\alpha(x)x^{2n} = \mu_{2n} > 0$ the odd moment vanish $\mu_{2n+1} = 0$, for n=0 we can take $\mu_0 = 1$, since in our case this 'Hamburger moment problem' is solvable (the Taylor series for the Xi-function is unique) for the measure $\omega(x) = d\alpha(x)$, the Hankel Matrix related to this moment problem defines a positive definite quadratic form

$$\sum_{i,j=0}^{n} \mu_{i+j}u_i\bar{u}_j = \int_{-\infty}^{\infty} d\alpha(x)\left(u_0 + u_1x + u_2x^2 + ... + u_nx^n\right)^2 \ge 0 \quad \text{with } u_i \in C \quad (1)$$

In the paper [1] Carlon and Gaston showed the Riemann Hypothesis equivalence in term of orthogonal Polynomials $\lim_{n\to\infty}\frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2+iz)}{\xi(1/2)}$,However they do not specify what the measure $\omega(x) = d\alpha(x)$ should be in order this formula to be true, our starting point is the taylor series for the Xi function defined in terms of the Riemann Zeta as $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$

$$\xi(s) = \sum_{n=0}^{\infty} a_{2n}\left(s-\frac{1}{2}\right)^{2n} \quad a_{2n} = \frac{4}{(2n)!}\int_1^{\infty} dx \frac{d\left(x^{3/2}\Psi'(x)\right)}{dx}\left(\frac{1}{2}\ln(x)\right)^{2n} x^{-1/4} \qquad (2)$$

This formula (taylor expansion) is valid for $|s| < \frac{1}{2}$, if we set $s = \frac{1}{2} + iz$ and divide all by $\xi(1/2)$ the taylor series becomes $\frac{\xi(1/2+iz)}{\xi(1/2)} = 1 + \sum_{n=1}^{\infty} \frac{a_{2n}}{a_0}(-1)^n z^{2n}$, the integrand inside (2) under a change of variable $x = e^{2u}$ can be considered as a 'Hamburger moment problem' we define

$$\frac{a_{2n}}{a_0}(2n)! = \mu_{2n} = \int_{-\infty}^{\infty} d\alpha(x) x^{2n} \quad \text{for even 'n'} \quad \text{and} \quad \mu_{2n+1} = 0 \qquad (3)$$

$$\omega(x) = d\alpha(x) = \begin{cases} \dfrac{1}{\xi(1/2)} \dfrac{d\left(e^{u} \dfrac{d\Psi(e^{2u})}{du} \right)}{du} e^{-u/2} & \text{if } x \geq 0 \quad \Psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x} \\ \omega(x) = \omega(-x) \end{cases} \qquad (4)$$

From this definition we can see that this measure is even and positive , in this case the 'Hamburger moment problem' can be solved by using the Taylor expansion of $\xi(1/2+iz)$ near z =0 , the odd moment are 0 as one would expect by symmetry arguments and the first moment is normalized to be $\mu_0 = 1$, this relation between 'Hamburger moment problem' and orthogonal Polynomials is important in order to avoid tedious calculations of the n-th order orthogonal polynomial $p_n(x)$, for example in general in order to obtain a orthogonal basis of Polynomials on a certain interval with respect to a given measure $d\alpha(x)$ one can use the set of non-orthogonal powers of x $\{1, x, x^2, x^3, ...\}$ and then compute $p_n(x)$ using the 'Gram-Schmidt' orthogonalization procedure

$$p_0(x) = 1 \qquad p_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n \mid p_k(x) \rangle}{\langle p_k(x) \mid p_k(x) \rangle} p_k(x) \qquad n \geq 1 \qquad (5)$$

Here the scalar product is defined via the Stieltjes integral $\langle f \mid g \rangle = \int_{-\infty}^{\infty} d\alpha(x) f(x) g(x)$

However this definition for $p_n(x)$ make these orthogonal polynomials hard to compute for practical calculations. A faster method to compute $p_n(x)$ without using the recurrence relation (5) is using the Determinant of a Hankell Matrix whose entries are precisely the moments (in our special case) $\frac{a_{2n}}{a_0}(2n)! = \mu_{2n} = \int_{-\infty}^{\infty} d\alpha(x) x^{2n}$, $\mu_{2n+1} = 0$ in this case we can compute the n-th orthogonal Polynomial as follows $p_n(x) = \frac{1}{\sqrt{D_n D_{n-1}}} Det\left(H_n \mid x\right)$ with:

$$p_n(x) = \frac{1}{\sqrt{D_n D_{n-1}}} \begin{vmatrix} 1 & 0 & \mu_2 & .... & \mu_n \\ 0 & \mu_2 & 0 & .... & \mu_{n+1} \\ .... & .... & ..... & .... & .... \\ \mu_{n-1} & \mu_n & \mu_{n+1} & ... & 0 \\ 1 & x & x^2 & .... & x^n \end{vmatrix} \qquad D_n = \begin{vmatrix} 1 & 0 & \mu_2 & .... & \mu_n \\ 0 & \mu_2 & 0 & .... & \mu_{n+1} \\ .... & .... & ..... & .... & .... \\ \mu_{n-1} & \mu_n & \mu_{n+1} & ... & 0 \\ \mu_n & \mu_{n+1} & \mu_{n+2} & .... & \mu_{2n} \end{vmatrix} \qquad (6)$$

For n=0 then $D_{-1} = 1$, (see reference [8]) , in our case $\mu_{2n+1} = 0$ and $\frac{a_{2n}}{a_0}(2n)! = \mu_{2n}$ this makes the Determinant easier to compute without using 'Gram-Schmidt' Orthogonalization for every positive 'n' , the relation to the conjecture proposed by Carlon and Gaston [1] is the fact that as $n \to \infty$ then Riemann Hypothesis is equivalent to $\lim_{n\to\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2+iz)}{\xi(1/2)}$, since all the roots of the orthogonal polynomials are real and distinct , and the polynomials must be odd or even functions, this hypothesis seems to be plausible unless there are multiple non-trivial zeros of the Riemann Zeta function $\zeta(\rho) = 0 = \zeta'(\rho)$ in this case the conjecture could be false but Riemann Hypothesis still remain true.

Ullman [9] has studied the functional relation between orthogonal polynomials $p_n(x)$ and the determinant of a certain Hankell Matrix , if we define $s_{2n}(x) = \sum_{k=0}^{2n-1} a_k z^{(k-1)}$ to be the n-th section of a Taylor power series then the distance $|s_{2n}(x) - p_{2n}(x)| \to 0$ as $n \to \infty$, Carlon and Gaston [1] studied the convergence of the Hadamard product for the Xi-function $\frac{\xi(1/2+iz)}{\xi(1/2)} = \prod_{j=0}^{\infty}\left(1 - \frac{z^2}{\gamma_j^{\,2}}\right)$, the idea is to associate the zeros of $p_{2n}(x)$,which we know to be real (and distincts) to these zeros of the Xi-function over the critical line $s = \frac{1}{2} + iz$, in this paper following their idea we have obtained an even measure $\omega(x) = d\alpha(x) = \omega(-x)$, this ansatz comes from the definitions (2) (3) (4) , from the definition of 'Hamburger moment problem' we cand find using the Borel resummation procedure the following relations (generating function for the moments μ_{2n} in terms of the Xi-function)

$$f(z) = \sum_{n=0}^{\infty} \frac{a_{2n}}{a_0}(-1)^n (2n)! z^{2n} = \int_0^{\infty} e^{-x} \frac{\xi(1/2+izx)}{\xi(1/2)} dx \quad \text{and} \quad f(z) = \int_{-\infty}^{\infty} \frac{d\alpha(x)}{1-(izx)^2} \qquad (7)$$

(Laplace transform) (Stietjles transform)

The argument for the formula (7) were obtained by us in a previous paper regarding divergent series [5] although we can prove (7) only in formal sense, in fact we have for the Laguerre or Legendre Polynomials with measure and moment $\{s_n = n! \ , \ \omega_1(x) = e^{-x}\}$ (Laguerre) and $\left\{s_{2n} = \frac{1}{2n+1} \ , \ s_{2n+1} = 0 \ , \ \omega_2(x) = 1\right\}$ (Legendre) , formula (7) can be stated as

$$\frac{e^{-1/z}}{z} E_1\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} n!(-z)^n = \int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{1+(zx)} \qquad \frac{\tan^{-1}(z)}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n} = \int_{-1}^{1} \frac{dx}{1+(zx)^2} \quad (8)$$

(Laguerre) (Legendre)

These results were already known , in the case of Laguerre Polynomials, the integral representation is just the Borel resummed series for the exponential integral , expression (8) can be used to extend the domain of the power series to every real or complex 'z' (unless we have poles) , formula (7) can be used in order to obtain some information about the moment μ_{2n} as $z \to \infty$ using

$$\int_{-\infty}^{\infty} \frac{d\alpha(x)}{x^2+z^2} \approx \sum_{k=0}^{\infty} \frac{(-1)^k \mu_{2k}}{z^{2k+2}} \qquad \frac{a_{2n}}{a_0}(2n)! = \mu_{2n} \quad \text{and} \quad \mu_{2n+1} = 0 \qquad (9)$$

As a final remark, we have constructed a set of orthogonal Polynomials $p_n(x)$ which are orthogonal with respect to a certain known measure $\int_{-\infty}^{\infty} d\alpha(x) p_n(x) p_m(x) = \delta_{m,n}$, the idea is if we could define a 'Rodrigues formula' $p_n(x) = \frac{1}{d_n \omega(x)} \frac{d^n}{dx^n}\left(\omega(x)\left(Q(x)\right)^n\right)$, where Q(x) is a function that does not depend on 'n' , if the conjecture $\lim_{n\to\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2+iz)}{\xi(1/2)}$ and Riemann Hypothesis with simple zeros is true , then using Cauchy's theorem we can give the following integral equation representation for Q(x) , $\frac{p_{2n}(0)}{\xi(1/2)} \frac{\xi(1/2+ix)}{d_{2n}^{-1}} \approx \frac{(2n)!}{2\pi i} \oint_C \frac{dz Q^{2n}(z)}{(z-x)^{2n+1}}$ here $\{d_n\}$ are some real constants and 'C' is a closed curve including the point z = x.

Although many of the results were stablished in Cardon and Gaston's paper he main difference with their paper is that they did not specify any measure $\omega(x)$ for the scalar product of function $\langle f \mid g \rangle$ involved in the 'Gram-Schmidt' procedure neither they did give any Hankel determinant representation for the n-th orthogonal Polynomial in terms of any known 'moment problem' , we have exploited the advantages of having a symmetryc measure $\omega(x) = \omega(-x)$ and the Hankel representation of orthogonal polynomials to simplify the Numerical calculations , we also have compared our results with other results referring Laguerre or Legendre Polynomials , our idea or conjecture is the following : the orthogonal polynomials $\{p_n(x)\}$ that Cardon and Gaston are looking for are related to the solvable moment problem $\int_{-\infty}^{\infty} d\alpha(x) x^{2n} = \mu_{2n} = \frac{a_{2n}}{a_0}(2n)! > 0$ (n even) , $\mu_{2n+1} = 0$ (n odd) $\omega(x) = d\alpha(x) = \omega(-x)$, one we have solved the problem and obtained the measure, we use the Hankel representation of the Polynomials $p_n(x) = \frac{1}{\sqrt{D_n D_{n-1}}} Det\left(H_n \mid x\right)$ to simplify the calculations to check if the conjecture in paper [1] is correct.

- *Relation between our given measure and the Fourier and Laplace integral for* $\xi(1/2+iz)$:

Although it could seem that we have chosen our measure at random in order to get the set $\{p_n(x)\}$, which is based only on the Taylor series expansion for the Xi-function ,we can also give a valid argument based on the Fourier (cosine) integral formula for $\Xi(z) = \xi\left(1/2+iz\right)$

$$\Xi(z) = 4\int_0^{\infty} du \Phi(u) \cos(uz) \quad \text{with} \quad \Phi(u) = \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{9u/2} - 3n^2\pi e^{5u/2}\right) e^{-n^2\pi e^{4u}} \quad (10)$$

The last expression inside (10) includes the first and second derivative of the Theta function $\Psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ evaluated at $x = e^{4u}$, to extend the defintion of Φ to negative values of 'u' we can impose the functional equation $\Phi(u) = \Phi(-u)$, we have the relation between our measure and the function involved in the Fourier Cosine transform (10) definition of $\xi(1/2+iz)$ as $\Phi\left(\frac{u}{2}\right) = A\omega(u)$, A is a real-valued constant that does not depend on the values of $\frac{a_{2n}}{a_0}$ or $(2n)!$.

We can reformulate our initial moment problem as $\int_{-\infty}^{\infty} dx\Phi(x)x^{2n} \propto \frac{a_{2n}}{a_0}\frac{(2n)!}{2^{2n}}$ in terms of $\Phi(u)$, using the Taylor series expansion for the cosine function $\cos(2zu) = \sum_{n=1}^{\infty}(-1)^n \frac{(zu)^{2n}2^{2n}}{(2n)!}$ and integration term by term, we recover the well-knwon Taylor expansion for Xi-function $\frac{\xi(1/2+iz)}{\xi(1/2)} = \sum_{n=0}^{\infty}(-1)^n \frac{a_{2n}z^{2n}}{a_0}$. Suprisingly this kind of relation also holds for the Chebyshev Polynomials (with a change of variable $x = \sin(t)$) and for the Legendre Polynomials

$$J_0(x) = \prod_n \left(1 - \frac{x^2}{\lambda_n^2}\right) = \frac{1}{\pi}\int_0^{\pi} dt \frac{cos(xt)}{\sqrt{1-t^2}} \qquad \frac{\sin(x)}{x} = \prod_n \left(1 - \frac{x^2}{n^2\pi^2}\right) = \frac{1}{2}\int_{-1}^{1} dt\cos(xt) \quad (11)$$

$J_0(\lambda_n) = 0 \quad n = 0,1,2,3,.....$ Real roots of zeroth-order Bessel function, as it can be seen the functions represented in (11) by a Fourier cosine transform with a certain measure , describe real function with ONLY real roots. This $\Phi(u)$ could be viewed as certain Probability distribution dF

$$\int_{-\infty}^{x} du\left(\sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{9u/2} - 3n^2\pi e^{5u/2}\right)e^{-n^2\pi e^{4u}}\right) = F(x) = \Pr ob[Y \le x] \quad (12)$$

or if we define a G function via $G_{-}(z-ix) + G_{+}(z+ix) = \cos(zx)$ and use the convolution definition $\left(G * dF\right)(z) = \int_{-\infty}^{\infty} G(z-ix)dF(x)$, then the Xi-function is proportional to the linear combination $\left(G_{+} * dF\right)(z) + \left(G_{-} * dF\right)(z)$, with $G_{\pm}(z \pm ix)$ defined on the upper/lower part of the complex plane .

Also In formula (7) we gave the expression for the moment generating function as the Laplace transform of the Xi function , now we extend by analytic continuation to define Laplace transform for complex values of 'z' via $z \to \pm si$ and use the definition of inverse Fourier transform , assuming $\frac{\xi(1/2+iz)}{\xi(1/2)}$ has only real zeros , then we can write $\frac{f(i/s) - f(-i/s)}{8i\pi s} = \Phi(s)$ valid for $s \in R^{+}$, $f(z) = \sum_{n=0}^{\infty}(-1)^n(2n)!\frac{a_{2n}z^{2n}}{a_0}$. An alternative method to invert the Laplace transform $\frac{f(1/z)}{z} = \int_0^{\infty} dxe^{-zx}\frac{\xi(1/2+ix)}{\xi(1/2)}$ Szego [8] is to define a set of secondary orthogonal polynomials via the integral $\int_{-\infty}^{\infty}\frac{p_{2n}(t) - p_{2n}(x)}{t^2 - x^2}\omega(t)dt = Q_{2n}(x)$, then the quotient (Pade approximant) $f_{2n}(z) = \frac{Q_{2n}(z)}{p_{2n}(z)}$ will converge to $\frac{f(i/z)}{z^2} = \int_{-\infty}^{\infty}\frac{d\alpha(x)}{z^2 - x^2}$, then if we expand into power series $-s\frac{Q_{2n}(is)}{p_{2n}(is)} = \sum_{i=0}^{\infty}(-1)^i c_{2i,2n}\frac{(2i)!}{s^{2i+1}}$ and perform a Laplace inverse transform to get the series $\sum_{i=0}^{\infty}(-1)^i c_{2i,2n}x^{2i}$, the conjecture by Cardon and Gaston is equivalent to $\lim_{n\to\infty} c_{2i,2n} = \frac{\mu_{2i}}{(2i)!}$ so both series $\sum_{i=0}^{\infty}(-1)^i \frac{a_{2i}}{a_0}\frac{1}{z^{2i}}$ and $\sum_{n=0}^{\infty}(-1)^i \frac{c_{2n,2i}}{z^{2i}}$ must converge (either in the usual sense for the sum or in the sense of the Borel resummation for the divergent series) to $\frac{\xi\left(\frac{1}{2}+\frac{i}{z}\right)}{\xi(1/2)} = z\int_0^{\infty} dxB(x)e^{-zx}$, (12)

$B(x) = \sum_{i=0}^{\infty}\frac{(-1)^{2i}}{(2i)!}\frac{a_{2i}}{a_0}\frac{1}{x^{2i}}$ in the limit $n \to \infty$, this method of continued fractions and Pade approximants applied to the moment problem is explained in [10] , expression (12) can be used to define the values of $\Xi(z) = \xi\left(1/2 + iz\right)$ for Real 'z' .

- *Probabilistic interpretation of the orthogonal polynomials* $p_n(x)$:

In Probability theory we have a 'Determinantal process' if the n-point correlation function of its Eigenvalues satisfy $\rho(x_1, x_2, ..., x_n) = \det\left[K(x_i, x_j)\right]_{i,j=1}^{n}$ for some integral Kernel operator 'K' with the symmetry $K(x,y) = K(y,x)$, this kind of determinantal process definition appears in Bornemann's [ ] evaluation of the Fredholm Determinant of an integral operator on the interval $\left(-\infty, \infty\right)$ defined by the expressions

$$d(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ..... \int_{-\infty}^{\infty} dx_n \det\left[K(x_i, x_j)\right]_{i,j=1}^{n} \qquad d(z) = \det\left[1 + zK\right] \quad (13)$$

With $u(x) + z\int_{-\infty}^{\infty} dyK(x,y)u(y) = f(x)$, and f a Real-valued function , Hilbert-Schmidt theory [11] for symmetric Kernel theory says that all the eigenvalues of the integral operator $\phi = z\int_{-\infty}^{\infty} dyK(y,x)\phi(y)$, are REAl , hence ALL the roots of the Fredholm determinant defined in (13) should be also real, Riemann Hypothesis would be then equivalent to the assertion

$$d(z) = \frac{\xi(1/2 + iz)}{\xi(1/2)} = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}(-1)^n \mu_{2n} \quad \text{, with d(z) defined on (13)} \qquad (14)$$

So the scaled Xi function is nothing but a 'Fredholm determinant' of certain integral operator with real Eigenvalues, if we could find such operator then we would have proved the Riemann Hypothesis, apparently there is a contradiction, since the Fredholm determinant includes powers of the form $(-z)^n$ and Riemann Xi-function definition involves powers of the form $(-1)^n z^{2n}$, this contradiction can be saved if we conjecture that the Xi-function (scaled) is nothing but the Fredholm determinant

$$\frac{\xi(1/2 + iz)}{\xi(1/2)} = d(z^2) = \det(1 + z^2 M.M^{\dagger}) = \det(1 + izM^{\dagger}).\det(1 - izM) \qquad (15)$$

And also we should have ONLY terms involving even powers of 'z' so in terms of Determinantal processes in probability and taking into account the definition of Fredholm determinant (13) in terms of the correlation functions , we should have

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ..... \int_{-\infty}^{\infty} dx_n \det\left[K(x_i, x_j)\right]_{i,j=1}^{n} \begin{cases} \mu_{2n+1} = 0 \text{ iff n is odd} \\ (-1)^n \mu_{2n} \text{ iff n is even} \end{cases} \qquad (16)$$

Here (16) is interpretated as the expected number of n-tuples , of Eigenvalues of the Kernel operator H(x,y) in $(-\infty,\infty)x(-\infty,\infty)x$.............. and so on , A possibility for the Kernel H(x,y) as an infinite-dimensional analogue to $M.M^{\dagger}$ is the 'iterated' kernel

$$H(x,y)=\int_{-\infty}^{\infty} du(i)K(x,u)(i)K(u,y) \quad i=\sqrt{-1} \qquad K(x,y)=K(y,x) \qquad (17)$$

An still open question would be , what is the Kernel $K(x,y)=K(y,x)$, or what it should be ? , since we are dealing with an Orthogonal Polynomial enssemble in Probability, then using the Christoffel-Darboux formula for the sum $K_{2N}(x,y)=\sum_{i=0}^{2N} p_i(x)p_i(y)p_i^{-2}(0)$ involving our set of orthogonal polynomials $\{p_n(x)\}$ with respect to the measure $\omega(x)$ defined in (4) in the limit $2N\to\infty$ (See Szego [8]) together with the Assumption of Cardon and Gaston for big n $\lim_{n\to\infty}\frac{p_{2n}(z)}{p_{2n}(0)}=\frac{\xi(1/2+iz)}{\xi(1/2)}$, the Kernel K(x,y) (up to some constant 'C' to make it Hermitian) is

$$K(x,y)=C\frac{\xi(1/2+ix)\xi'(1/2+iy)-\xi'(1/2+ix)\xi(1/2+iy)}{x-y} \qquad (18)$$

And using the defintion for the first 'iterated' Kernel (with the integral assumed to exist in Cauchy's principal value) obtained in (17) we can obtain the n-th iterated Kernel for the integral operator and calculate $\det\left[H(x_i,x_j)\right]_{i,j=1}^{n}$ or $\det\left[K(x_i,x_j)\right]_{i,j=1}^{n}$ for every 'n'.

CONCLUSIONS AND FINAL REMARKS

Inspired by a previous idea based on a formulation for Riemann Hypothesis as the limit $\lim_{n\to\infty}\frac{p_{2n}(z)}{p_{2n}(0)}=\frac{\xi(1/2+iz)}{\xi(1/2)}$ and using the coefficients for the Taylor series of Xi-function $\frac{\xi(1/2+iz)}{\xi(1/2)}=\sum_{n=0}^{\infty}(-1)^n\frac{a_{2n}}{a_0}z^{2n}$ we have defined a 'Hamburger moment problem' based on the measure $\frac{1}{\xi(1/2)}\frac{d\left(e^u\partial_u\Psi(e^{2u})\right)}{du}e^{-u/2}=\omega(x)$, $\Psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$ for $x\ge 0$, this definition can be extended to negative values by using the functional equation $\omega(x)=\omega(-x)$,with the solution to this moment problem we construct a set of orthogonal Polynomials { $p_n(x)$ } ,for even big 'n' the Hankel determinant $p_{2n}(x)=\frac{1}{\sqrt{D_{2n}D_{2n-1}}}Det\left(H_{2n}\,|\,x\right)$ should approach to the Riemann Xi-function divided by $\xi(1/2)=a_0$ as Cardon and Gaston deduced , from the measure $\omega(x)$ we can give the (asymptotic) series representation for $\int_0^{\infty}\frac{d\alpha(x)}{x^2+z^2}\approx\sum_{k=0}^{\infty}\frac{(-1)^k\mu_{2k}}{2z^{2k+2}}=\frac{1}{2z^2}f\left(\frac{1}{z}\right)$, in case $\lim_{n\to\infty}\frac{p_{2n}(z)}{p_{2n}(0)}=\frac{\xi(1/2+iz)}{\xi(1/2)}$ holds , then Riemann Hypothesis would follow from the fact that the the infinite product $\prod_{j=0}^{\infty}\left(1-\frac{z^2}{\gamma_j^{\ 2}}\right)$ and the determinant $Det\left(H_{2n}\,|\,x\right)$ would have the same set of (real) roots $\gamma_j \in R$, so $p_{2n}(\gamma)=0=\xi\left(\frac{1}{2}+i\gamma\right)\ \gamma\in R \quad n\to\infty$, Numerical computations can be simplified by the fact that for a fixed '2n' we can use the determinants of Hankel matrices to compute $p_{2n}(x)$. We also discuss another possible interesting viewpoints and the connection of our set of Orthogonal Polynomials with some concepts of Probability theory and Integral operators, specially a possible interpretation of the Xi-function as the Fredholm determinant of an operator $\frac{\xi(1/2+iz)}{\xi(1/2)}=\det\left[1+zMM^{\dagger}\right]$,whose coefficients are related to the n-point correlation function of a certain determinantal process.

References:

[1] Cardon David and Gaston Sharleen "*An equivalence for the Riemann Hypothesisin terms of Orthogonal Polynomials* "Journal of approximation theory, ISSN 0021-9045, Vol. 138, Nº 1, 2006

[2] Bender Carl and Ben-Naim E. "*Nonlinear-integral-equation Construction of Orthogonal Polynomials*" Journal of Nonlinear Mathematical Physics , Vol 15 (2008) 73-80.

[3] Bornemann F. "*On the Numerical evaluation of Fredholm determinats"* Mathematics of Computation (2009) ISSN 0025-5718

[4] Conrey, J. B. "*The Riemann Hypothesis*." *Not. Amer. Math. Soc.* **50**, 341-353, 2003

[5] Garcia J.J ; "*A comment on mathematical methods to deal with divergent series and integrals"* e-print avaliable at http://www.wbabin.net/science/moreta10.pdf

[6] Keiper, J. B. "*Power Series Expansions of Riemann's Xi-Function*." *Math. Comput.* **58**, 1992.

[7] Shohat, J. A.; Tamarkin, J. D. (1943), "*The Problem of Moments"*, New York: American mathematical society, ISBN 0821815016.

[8] Szego Gabor " *Orthogonal Polynomials"* American Mathematical Society **ISBN-10:** 0821810235

[9] Ullman J. "*Hankel determinants whose elements are sections of a Taylor series. Part I* " Duke Math. J. Volume 18, Number 3 (1951), 751-756.

[10] Wall H.S (1948). "*Analytic Theory of Continued Fractions"* . D. Van Nostrand Company Inc.

[11] Weber H. And Arfker G. "*Mathematical Methods for Physicists"*. Harcourt/Academic Press (2000)

[12] Yuval P ,Bálint V. Manjunath K., And J. Ben Hough, "*Determinantal process and Independence"* Probab. Surveys Volume 3 (2006), 206-229.