

ZETA REGULARIZATION APPLIED TO THE PROBLEM OF RIEMANN HYPOTHESIS AND THE CALCULATION OF DIVERGENT INTEGRALS

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ABSTRACT: In this paper we review some results of our previous papers involving Riemann Hypothesis in the sense of Operator theory (Hilbert-Polya approach) and the application of the negative values of the Zeta function $\zeta(1-s)$ to the divergent integrals $\int_0^{\infty} x^{s-1} dx$ and to the problem of defining a consistent product of distributions of the form $D^m \delta(x) D^n \delta(x)$, in this paper we present new results of how the sums over the non-trivial zeros of the zeta function $\sum_{\rho} h(\rho)$ can be related to the Mangoldt function $\Psi_0(x)$ assuming Riemann Hypothesis.

- *Keywords:* Zeta regularization, Urysohn equation, exponential nonlinearity, Riemann Hypothesis Hilbert-Polya operator, divergent integral

1. Spectral Zeta function $\zeta_H(s)$ and Riemann Hypothesis :

In case Riemann Hypothesis (RH) is true, in a previous paper [6] we give the physical equivalence between the explicit formula for the Chebyshev function $\Psi_0(x)$ and the formula for the trace of the Unitary operator $\hat{U} = e^{i\hat{H}}$, where H is the Hamiltonian operator $\zeta\left(\frac{1}{2} + i\hat{H}\right)|\phi_n\rangle = 0$, that is H is precisely the Hilbert-Polya operator solution to Riemann Hypothesis, let be the integral representation

$$\Psi_0(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} = \begin{cases} x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1-x^{-2}) & x > 1 \\ 0 & x < 1 \end{cases} \quad (1.1)$$

Letting $x = e^u$, and differentiating with respect to 'u' we find the (trace) identity

$$e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} = \sum_{n=-\infty}^{\infty} e^{iuE_n} = \text{Tr}\{\hat{U} = e^{iu\hat{H}}\} \quad (1.2)$$

Using the semiclassical representation for the trace $\sum_{n=-\infty}^{\infty} e^{iuE_n}$ in terms of an integral over Phase Space, we have that the potential $V(x)$ inside Hamiltonian H can not be arbitrary but must satisfy a kind of nonlinear Urysohn integral equation

$$\int_{-\infty}^{\infty} r^{iV(x)} dx = \sqrt{\frac{\log(r)}{\pi r^{-1}}} \left(1 - \frac{d\Psi_0(r)}{dr} - \frac{1}{r^3 - r} \right) e^{-i\pi/4} \quad r = e^u \quad (1.3)$$

The derivative of the Chebyshev function is defined as $\frac{d\Psi_0(x)}{dx} \frac{1}{\log(x)} = \sum_{p,p'} \delta(x - p^v)$

(sum taken over prime and prime powers). However (1.3) is too complex to have a known analytic solution, a good method to solve would be to suppose that the Operator proposed by Berry and Keating [2] plus an interaction is the correct Hilbert-Polya operator, in that case $H_b = xp + \alpha W(x)$ and we can linearize (1.3) at first order in the coupling constant ' α ' as

$$\text{Tr}\{e^{iu\hat{H}}\} - \frac{2\pi}{|u|} \approx iu\alpha \int_{-\infty}^{\infty} dp \hat{F}\{W(x), u\} \quad \hat{F}\{W(x), u\} = \int_{-\infty}^{\infty} dx e^{iuxp} W(x) \quad (1.4)$$

Also, if we introduce the function $Z_u(\eta) = \int_{-\infty}^{\infty} dx e^{iu(V(x)+x\eta)}$, with continuous partial derivatives $\partial_{\eta}^k Z_u(\eta)$, then solving (1.3) is equivalent to finding a solution to the initial-value problem

$$Z_u(\eta) + \eta \frac{\partial Z_u(\eta)}{\partial \eta} + iu \left(\sum_{k=0}^{\infty} d_k \frac{k}{(iu)^k} \frac{\partial^k}{\partial \eta^k} \right) Z_u(\eta) = 0 \quad (1.5)$$

$$\left(e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \right) \sqrt{\frac{u}{\pi}} e^{-i\pi/4} = Z_u(0)$$

Expression (1.8) tells us that proving RH is equivalent to show that the ODE given in (1.5) with $\{d_k\} \in \mathbb{R}$ and $d_k = \frac{1}{k!} \frac{d^k V(x)}{dx^k} \Big|_{x=0}$, $V(x) = \sum_{k=0}^{\infty} d_k x^k$ using (1.5) together with a finite power expansion for $V(x)$, using (1.5) we could obtain the constants $\{d_k\} \in \mathbb{R}$ to get an approximate solution for the potential $V(x)$.

If RH is true and $\zeta\left(\frac{1}{2} + iE_n\right) = 0$, with $E_n = -E_{-n}$ being the eigenvalues of a certain operator $H = p^2 + V(x)$, using expression (1.2) and the functional equation $\zeta(1-s) = 2(2\pi)^{-s} \text{Cos}\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$, then for $n \geq 0$ we can define an spectral Zeta function, involving the nontrivial zeros of Zeta and primes and prime powers

$$\sum_{n=0}^{\infty} \frac{1}{E_n^s} = \frac{\text{Sec}\left(\frac{\pi s}{2}\right)}{2\Gamma(s)} \int_0^{\infty} dt \text{Tr}\left\{e^{it\hat{H}}\right\} = (2\pi)^{-s} \frac{\zeta(s)}{\zeta(1-s)} \int_0^{\infty} dt e^{t/2} \left(1 - e^{-t} \frac{d\Psi_0(e^t)}{dx} - \frac{1}{e^{3t} - e^t}\right) t^{s-1} \quad (1.6)$$

The value $\prod_{n=0}^{\infty} E_n = e^{-\frac{d\zeta(0)}{ds}}$ would be the regularized product of all the positive ‘Eigenvalues’ $\{E_n\}$ this expression can also be used to obtain a Dirac measure for the E_n , let us introduce

$$\sum_{n=0}^{\infty} E_n^{-s} = \int_0^{\infty} dt \left(\sum_{n=0}^{\infty} \frac{1}{E_n} \delta\left(t - \frac{1}{E_n}\right) \right) t^{s-1} \quad \frac{\zeta(s)}{\zeta(1-s)} = \int_0^{\infty} dt K_0(t) t^{s-1} \quad (1.7)$$

Using the properties of the Mellin transform applied to solve linear integral operators $I[f] = \int_0^{\infty} dt R(xt) f(t)$, if we combine (1.6) and (1.7) we get the result

$$\Theta(x) = \sum_{n=0}^{\infty} \frac{1}{E_n} \delta\left(x - \frac{1}{E_n}\right) = \int_0^{\infty} \frac{dt}{t} K_0(2\pi xt) \left(e^{1/2t} + \frac{e^{-1/2t}}{t^2} \frac{d\Psi_0(e^{1/t})}{dt} - \frac{e^{1/2t}}{e^{3/t} - e^{1/t}} \right) \quad (1.8)$$

If we took the Mellin transform $\int_0^{\infty} dx x^{s-1}$ inside (1.8) together with the change of variable $xt=z$ we would recover equation (1.6), note that the Mellin transform of the Kernel $K_0(2\pi xt)$ does not depend on the nontrivial zeros $\rho = \frac{1}{2} + it$.

Using test functions $\frac{1}{x} h\left(\frac{1}{2} \pm \frac{i}{x}\right)$ inside (1.8) obtained from our Trace formula for $\text{Tr}\left\{e^{in\hat{H}}\right\}$ we can relate the convergent sum $\sum_{\rho} h(\rho)$ to a sum over primes and prime powers

$$\int_0^1 dx h\left(\frac{1}{2} + \frac{i}{x}\right) \int_0^\infty dt \frac{K_0(2\pi xt)}{xt} \left(e^{1/2t} + \frac{e^{-1/2t}}{t^2} \frac{d\Psi_0(e^{1/t})}{dt} - \frac{e^{1/2t}}{e^{3/t} - e^{1/t}} \right) + c.c = \sum_\rho h(\rho) \quad (1.9)$$

Formula (1.9) and its result can be compared with sums $\sum_\rho \frac{a^\rho}{\rho}$ (explicit formula for Chebyshev function) and $Z(n) = \sum_\rho \frac{1}{\rho^n}$ $n \in \mathbf{N}$, that can be calculated exactly.

2. Zeta regularization for divergent integrals:

Given the function $f(x) = x^m$, we can use the Euler-Maclaurin summation formula to obtain a recurrence relation between an integral of the form $I(m, \Lambda) = \int_0^\Lambda p^m dp$ $m \in \mathbf{Z}^+$ with $m \int_0^\Lambda x^{m-1} dx = \Lambda^m$ and the series $\sum_{i=0}^{\Lambda-1} i^m$, ref [7]

$$I(m, \Lambda) = (m/2)I(m-1, \Lambda) + \sum_{i=0}^{\Lambda-1} i^m - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} a_{mr} (m-2r+1) I(m-2r, \Lambda) \quad (2.1)$$

The coefficients $a_{mr} = \frac{\Gamma(m+1)}{\Gamma(m-2r+2)}$ vanish when $m+2 \leq 2r$, hence the sum in (2.1) is finite if m is an integer, in the physical limit the cutoff Λ must be taken to infinite, this makes the series $\sum_{i=0}^{\Lambda-1} i^m$ to be divergent for $m \geq -1$, in this case we should use the Functional equation for the Zeta function to obtain the value

$$\lim_{\Lambda \rightarrow \infty} \sum_{n=1}^{\Lambda-1} n^m = 1 + 2^m + 3^m + \dots + \Lambda^m \rightarrow \zeta_R(-m) = \zeta(-m) \quad (2.2)$$

(2.2) is the Zeta-regularized value for the divergent sum involved in (2.1), using this method we can compute the divergent integrals $I(m, \Lambda)$ $\Lambda \rightarrow \infty$, for $m=1,2,3$

$$I(0, \Lambda) = \zeta(0) = -1/2$$

$$I(1, \Lambda) = \frac{I(0, \Lambda)}{2} + \zeta(-1) \tag{2.3}$$

$$I(2, \Lambda) = \left(I(0, \Lambda) \frac{1}{2} + \zeta(-1) \right) - \frac{B_2}{2} a_{21} I(0, \Lambda)$$

$$I(3, \Lambda) = \frac{3}{2} \left(\frac{1}{2} (I(0, \Lambda) + \zeta(-1)) - \frac{B_2}{2} a_{21} I(0, \Lambda) \right) + \zeta(-3) - B_2 a_{31} I(0, \Lambda)$$

The case $m=0$ is just the divergent series $1+1+1+1+1+1+1+1+1+\dots$ taking the regularized value $-1/2$.

For an arbitrary function $f(x)$ so its integral would diverge as a power of the cutoff Λ^{N+1} we could expand $f(x)$ into a Laurent series convergent for $|x| < 1$ and $|x| > 1$ so we find

$$\int_a^\Lambda dx f(x) = \sum_{r=0}^N c_r I(r, \Lambda) + c_{-1} I(a, -1, \Lambda) + O(\Lambda^{-1}) - \sum_{i=0}^N \int_0^a dx (c_i x^i) - \sum_{j=2}^{\infty} c_{-j} a^{-j} \tag{2.4}$$

$\{c_i\} \in \mathbb{R}$, taking $\Lambda \rightarrow \infty$, and using (2.1) (2.2) (2.3) to regularize the divergent

integrals $I(m, \Lambda)$ we could obtain a regularized value for the integral $\int_0^\Lambda dx f(x)$,

however the logarithmic divergent integral $I(a, -1, \Lambda) = \int_a^\Lambda \frac{dx}{x}$ can not regularized by our

formulae, the solution would be to use the Euler-Maclaurin summation to approximate the divergent integral by a divergent Harmonic sum that can be attached a ‘Ramanujan

sum’ $\gamma - \sum_{n=1}^a \frac{1}{n}$ (γ =Euler-Mascheroni constant)

o *Zeta regularized product of distributions:*

Formulae (2.1-2.3) can be used to compute divergent integrals of the form $\int_0^\infty x^{s-1} dx$, but

also could give an answer to the problem of multiplication of two distributions involving Dirac delta and its derivatives $D^m \delta(x)$, if we tried to define the product of distributions involving delta functions we could use the ‘convolution theorem’ applied to the Fourier transform (A =normalization constant):

$$(2\pi)^2 i^{m+n} D^m \delta(\omega) D^n \delta(\omega) = F_\omega (x^m * x^n) = AF_\omega \left\{ \int_{-\infty}^{\infty} dt t^m (x-t)^n \right\} \tag{2.5}$$

Unfortunately (2.5) makes no sense , the integral is divergent for every real or complex value of ‘x’ , if m and n are positive integres using the Binomial theorem

$$i^{m+n} D^m \delta(\omega) D^n \delta(\omega) = \sum_{k=0}^n \binom{n}{k} i^{m+k} A D^{n-k} \delta(\omega) (-1)^k i^{n-k} D^{m+k} \delta(0)^{[R]} \quad (2.6)$$

$$i^{m+n} D^m \delta(\omega) D^n \delta(\omega) = \sum_{k=0}^n \binom{n}{k} i^{m+k} A D^{n-k} \delta(\omega) (-1)^k i^{n-k} \left((-1)^{m+k} + 1 \right) \int_0^{\infty} x^{m+k} dx \quad (2.7)$$

‘R’ stands for regularization, taking the divergent integrals for the dirac delta and its derivatives evaluated at x=0 , which is proportional to $\int_{-\infty}^{\infty} x^k dx$ for k=2r+1 (Odd) the integral is 0 , for k=2r the integral can be written as $i^{2r} D^{2r} \delta(0) = 2I(2r, \Lambda) \quad \Lambda \rightarrow \infty$ (r=integer) and can be evaluated using (2.1) and (2.2) .

The expression (2.7) is real ,this is what one would expect since the product of two distributions taking only real values must be real , however (2.6) is not still invariant under the change $m \rightarrow n$ and $n \rightarrow m$ (this is a mistake we made in paper [7]) so we should take a more symmetrical product of distributions defined by

$$\left(D^m \delta \otimes D^n \delta \right)_R (\omega) = \frac{1}{2} \left(D^m \delta(\omega) D^n \delta(\omega) + D^n \delta(\omega) D^m \delta(\omega) \right) \quad (2.8)$$

The simplest case is m=n=0 so $(\delta \otimes \delta)_R (\omega) = -A \delta(\omega)$

For the case of ‘m’ and ‘n’ not being an integer or we have a shifted dirac delta $D^k \delta(x-a)$, we could use the identities for the k-th power of ‘x’ or the traslation operator e^D and $D = \frac{d}{dx}$ in the form

$$e^{-aD} D^r \delta(x) = \sum_{j=0}^{\infty} (-1)^j \frac{a^j}{j!} D^{r+j} \delta(x) = \delta(x-a) \quad D^r = \sum_{k=0}^{\infty} \binom{r}{k} (D-1)^k \quad (2.9)$$

In case of integrals on $R^d \int_{R^d} dk F(\vec{k})$, if the function F , is invariant under Lorentz transformations, then making a Wick rotation to imaginary time $t \rightarrow it$, then in 4-dimensional polar coordinates our integral can be evaluated as $\int d\Omega \int_0^{\infty} dr f(r) r^{d-1}$, if not we could replace the integral over $d\Omega$ by a discrete sum $\sum_i \int_0^{\infty} dr f(r, \Omega_i) r^{d-1}$

Appendix A: an integral Trace for the Green function

A formula for the sum $\sum_{n=0}^{\infty} \delta(E - E_n)$ in terms of the Trace of the 'Resolvent' (green function) of a Quantum Hamiltonian $\hat{H}\phi_n = E_n\phi_n$ can be defined as:

$$\text{Tr}\{G(x, x', E)\} = \int_{R^4} d^4x G(x, x, E) = \pi \sum_{n=0}^{\infty} \delta(E - E_n) \quad G(x, x', E) = \frac{1}{E + i\varepsilon - \hat{H}} \quad (\text{A.1})$$

One of the easiest method to prove this, is to consider that given a convergent series with sum S and its Borel transform B(s) defined by $B(S, a_n) = \int_0^{\infty} dt \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \right) e^{-t}$ then

$S=B(S)$, $S = \sum_{n=0}^{\infty} a_n$ in this case if we take the series

$$\frac{1}{E + i\varepsilon - H} = \frac{E^{-1}}{1 + \frac{i\varepsilon - H}{E}} = E^{-1} \left(\sum_{n=0}^{\infty} \frac{(-1)^n (i\varepsilon - H)^n x^n}{E^n n!} \right) = \int_0^{\infty} dt e^{-t(1+i\varepsilon-\hat{H})} \quad (\text{A.2})$$

Where ε is an small number so $\varepsilon \rightarrow 0$, then using the formula for the Principal value $P.V\left(\frac{1}{x}\right) = \pi\delta(x) + \frac{1}{x+i\varepsilon}$, in this case taking the trace of the operator inside (A.2) we can give a proof to (A.1) using the technique of Borel resummation.

Another example of the method of Borel resummation, let be $P(x) = \sum_{n=0}^{\infty} (-1)^n \alpha(n) x^n$ the generating function of the coefficients $\alpha(n)$, let be the function f(t) defined by

$\alpha(s-1) = \int_0^{\infty} dt f(t) t^{s-1}$ then using again the Borel-generalized resummation

$$P(x) = \int_0^{\infty} dt \left(\sum_{n=0}^{\infty} (-1)^n (xt)^n \right) f(t) = \sum_{n=0}^{\infty} (-1)^n \alpha(n) x^n \quad \text{or} \quad P(x) = \int_0^{\infty} dt \frac{f(t)}{1+xt} \quad (\text{A.3})$$

If we took the Mellin transform on both sides $\int_0^{\infty} dx x^{s-1}$ we Would find

$\hat{P}(s) = \hat{K}(s)\hat{F}(1-s)$, or in terms of improper integrals

$$\int_0^{\infty} dt \left(\sum_{n=0}^{\infty} \alpha(n) (-x)^n \right) = \frac{\pi\alpha(-s)}{\sin(\pi s)} \quad \text{since} \quad \int_0^{\infty} dt f(t) t^{-s} = \alpha(-s) \quad (\text{A.5})$$

This last formula is known as ‘Ramanujan Master theorem’ , note that we have proved this only using the fact that for a convergent series its sums and Borel transform must be equal $S=B(S)$.

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