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EULER-MACLAURIN SUM FORMULA AND ITS GENERALIZATIONS AND APPLICATIONS

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Addres: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

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ABSTRACT: We study several possible generalizations of the Euler-Maclaurin formula, for several variables and infinite-dimensional spaces and its possible applications to Number theory and other branches of mathematics. Also we study how using this Euler-Maclaurin summation we can provide an approximate differential equation whose solution in the sense of distribution theory is just the Mertens function M(x) or the sum of Liouville function λ(n).

- *Keywords:=* Euler-Maclaurin sum formula, distributions

EULER-MACLAURIN SUM FORMULA AND ITS POSSIBLE GENERALIZATIONS AND APPLICATIONS

The Euler-MacLaurin sum formula relating the sum of an infinite series to a definite integral over the interval $(0,\infty)$ in the form:

$$\frac{1}{2}\left(f(0)+f(\infty)\right)+\sum_{n=1}^{\infty}f(n)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\left(D^{2r-1}f(\infty)-D^{2r-1}(0)\right)=\int_{0}^{\infty}dxf(x) \qquad (1)$$

If the function f(x) and its derivatives vanishes as $x\to\infty$, the formula (1) can be used to obtain approximate results for the infinite series on the left, for other cases we can make the replacement $D^{k}\to h^{k}D^{k}$ and multiply the whole left term by ‘h’ (step) so the infinite series is now $\sum_{n=1}^{\infty}f(nh)$, the most straight proof for the Euler-Maclaurin summation formula is to use the identity:

$$-\left(1+e^{D}+e^{2D}+............\right)=\frac{1}{D}.\frac{D}{e^{D}-1}=\frac{1}{D}-\frac{I}{2}+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}D^{2r-1} \qquad (2)$$

I is the identity operator , $D=\frac{d}{dx}$ with $e^{D}f(x)=f(x+1)$, formula (2) provides another useful information since we could replace 'D' by any differential or integral operator , for example for R^{n} if we define the scalar $\xi=\left(\vec{k}.\vec{\nabla}\right)$ $\mathrm{k}=(a_1,a_2,....,a_n)$ and $\vec{\nabla}$ is the Gradient , if we insert ξ inside (2) equation with the following property $e^{\vec{k}.\vec{\nabla}}f(\vec{r})=f(\vec{r}+\vec{k})$ with $\vec{r},\vec{k}\in R^{n}$, we have

$$\frac{1}{2}f(\vec{0})+\sum_{n=1}^{\infty}f(\vec{0}+n\vec{k})+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\left(\vec{k}.\vec{\nabla}\right)^{2r-1}f(\vec{0})=-\left(\vec{k}.\vec{\nabla}\right)^{-1}f(\vec{0}) \qquad (3)$$

the notation $\vec{0}$ is the vector whose components are all '0' , however if we define an initial vector 'r_0 ' the formula (3) still holds even for infinite-dimensional spaces, the notation $\left(\vec{k}.\vec{\nabla}\right)^{-1}f(\vec{r})$ $\vec{r}=\left(x_1,x_2,.....,x_n\right)\in R^{n}$ means that we should solve the linear PDE $\sum_{i=1}^{n}a_i.\frac{\partial g}{\partial r_i}=f(\vec{r})$ in absence of further boundary or initial value problem and if the multivariable Fourier transform for 'f' exists then

$$\left(\vec{k}.\vec{\nabla}\right)^{-1}f=\frac{-i}{\left(2\pi\right)^{n}}\int d^{n}\omega e^{i\vec{\omega}.\vec{r}}F(\vec{\omega})\left(\vec{k}.\vec{\omega}\right)^{-1} \quad \text{F(w) = Fourier transform of 'f'} \qquad (4)$$

certainly if we consider that a function is a vector that belongs to a functional space and use the next generalizations from usual derivatives to Functional ones:

$$\sum_{i=1}^{n}dx_i.\frac{\partial}{\partial x_i}\to\int_{R^{n}}d^{n}x\frac{\delta}{\delta\phi} \qquad \delta_i^{j}\to\delta(i-j) \qquad \frac{\delta F}{\delta\phi}=\lim_{\varepsilon\to 0}\frac{F[\phi+\varepsilon\delta(x-y)]-F[\phi]}{\varepsilon} \qquad (5)$$

then if we use as a vector 'k' , the one whose components are $k_i=\delta_i^{u}$ (Kronecker's delta) for a fixed value 'u' formula (3) for Functional (Path) integrals reads

$$\int_{\phi_0}D[\phi]F[\phi]=\frac{F[\phi_0]}{2}h+h\sum_{n=1}^{\infty}F[\phi_0+nh\delta(x-u)]+R \qquad (6)$$

$$R=\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}h^{2r}\int dx_1\int dx_2...........\int dx_{2r-1}\frac{\delta^{2r-1}F[\phi]}{\delta\phi(x_1)\delta\phi(x_2)......\delta\phi(x_{2r-1})} \qquad (7)$$

again 'h' is an step of the series (6) and the functional derivatives $\frac{\delta^{2r-1}F[\phi]}{\delta\phi(x_1)\delta\phi(x_2)......\delta\phi(x_{2r-1})}$ should be evaluated at the 'point' $\phi=\phi_0$, note that the expression for the Remainder 'R' is in general a distribution and will need to be regularized , the expression (6) is the Euler-Maclaurin summation formula for functional spaces and path integrals , the integral in (6) will be taken over all the possible functions and can be considered an infinite dimensional generalization of the multiple integral:

$$\int_{\phi_0} D[\phi]F[\phi] \leftarrow \int_{c_1}^{\infty} dx_1 \int_{c_2}^{\infty} dx_2 \int_{c_3}^{\infty} dx_3 ....\int_{c_n}^{\infty} dx_n F(x_1, x_2, ..., x_n) \qquad n \to \infty \tag{8}$$

and $\phi_0(u_j) = c_j$ for j=1,2,3,4,…..formula (6) can be used to define some kind of Riemann sum for the path integral formalism (although due to the delta derivative in case functional are non-linear we would have some problems with distribution theory) another kind of approach to path integral is the Poisson summation formula

$$\int_{\phi_0} D[\phi] \sum_{m=-\infty}^{\infty} e^{iS[\phi]+2\pi i m \varepsilon^{-1}\langle 1|\phi+\phi_0\rangle} = \sum_{m=-\infty}^{\infty} e^{iS[\phi_0+m\delta(x-y)]}\varepsilon \qquad (\varepsilon = \text{step}) \qquad W[\phi] = \exp\left(i\int \phi dx\right) \tag{9}$$

with the ‘periodic’ functional $W[\phi] = W[\phi + 2\pi\delta(x-y)]$, here $S[\phi]$ is classical action of the system and $D[\phi] = D_0[\phi]M[\phi]$ with ‘M’ a functional and $D_0[\phi]$ a traslational invariant measure that remains invariant under the change $\tilde{\phi}(x) = \phi(x) + k\delta(x-y)$ if M is different from ‘1’ we should replace the action inside the exponential by the expression involving the logarithm of M as $S[\phi] - i\ln M[\phi]$, for other kind of Functional integral we may use the identity for Functional integrals

$$\int_{\phi_1} D[\phi] e^{iS[\phi]} - \int_{\phi_0} D[\phi].e^{iS[\phi]} = \int_{\phi_0}^{\phi_1} D[\phi] e^{iS[\phi]} \qquad S[\phi] = \int dx.L(\phi, \partial_\mu \phi) \tag{10}$$

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- *Euler sum formula and a differential identity for the Mertens and Liouville functions:*

The main application of Euler-Maclaurin formula is to obtain the approximate sum of infinite series, if f(x) and all its derivatives vanish as $x \to \infty$ then (1) gives us the approximate sum of the infinite series if we know how to obtain the integral $\int_0^\infty dt f(t)$, in many other cases if we knew an approximate asymptotic expression of an integral

$$\int_x^\infty dt g(t) \approx h(x)\left(a_0(x) + \frac{a_1(x)}{x} + \frac{a_2(x)}{x^2} + ...\right) \qquad \lim_{x\to\infty} a_i(x) = 0 \quad \forall i \tag{11}$$

then from (1) and (11) we could get an asymptotic expansion for series $\sum_{n=x}^{\infty} f(t)$.

Another important property of Euler-Maclaurin sum formula is that in case m is an integer m >0 , it gives a link between the zeta regularized sum of the divergent series $\zeta(-m)_R = \sum_{i=1}^{\infty} i^m$ and the divergent integral $\int_0^\Lambda dx x^m$ with $\Lambda \to \infty$ in the form

$$I(m,\Lambda)=(m/2)I(m-1,\Lambda)+\zeta(-m)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}a_{mr}(m-2r+1)I(m-2r,\Lambda) \qquad (12)$$

Ref. [3], $a_{mr}=\frac{\Gamma(m+1)}{\Gamma(m-2r+2)}$, these coefficients are non-zero unless $r\ge\frac{1}{2}(m+2)$,hence the recurrence in (12) is finite if m is an integer and $\zeta(-2u)=0$ for u=1,2,3,4,…….

Euler-Maclaurin sum formula can be extended to include distributions, if we choose a suitable set of test function so $\varphi(x)\in C^{\infty}$ (infinitely differentiable functions) with a convergent sum $\sum_{n=0}^{\infty}\varphi(x-n)$ for every $x\in(-\infty,\infty)$ and any distribution M(x)

$$\sum_{n=1}^{\infty}\langle M\,|\,\varphi(x-n)\rangle=\langle M\,|\,\partial^{-1}\varphi(x)\rangle+\frac{\langle M\,|\,\varphi(x)\rangle}{2}+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\langle M\,|\,\partial^{2r-1}\varphi(x)\rangle \qquad (13)$$

M=M(x) is a distribution , $\langle f\,|\,g\rangle$ is the scalar product on the interval $(-\infty,\infty)$, so (13) is the generalization of identity (2) to include distribution theory , with the assumption that the identity $e^{aD}M(x)=M(x+a)$ still holds whenever M is a distribution.

If we consider Euler-Maclaurin sum formula expressed in the form

$$-\frac{f(x)}{2}+\sum_{n=1}^{\infty}f\left(\frac{x}{n}\right)+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\frac{\partial^{2r-1}}{\partial t^{2r-1}}f\left(\frac{x}{t+1}\right)_{t=0}=\int_{0}^{\infty}dtf\left(\frac{x}{t+1}\right) \qquad (14)$$

'x' is a fixed parameter and $\lim_{t\to\infty}f\left(\frac{x}{t}\right)=0$, expression (13) yields to a differintegral equation in variable x , to explore its utility let's consider the next integral identities

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\frac{1}{s}\frac{x^{s}}{\zeta(s)}=\sum_{n\le x}\mu(n)=M(x) \qquad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\frac{1}{s}\frac{x^{s}}{\zeta(s)}=\sum_{n\le x}\lambda(n) \qquad (15)$$

M(x) is the Mertens function and $\lambda(x)$ is Liouville function (see ref. [1]) , Let's suppose we want to study the solution of the differential-integral equation

$$\frac{f(x)}{2}-\sum_{r=1}^{N}\frac{B_{2r}}{(2r)!}\frac{\partial^{2r-1}}{\partial t^{2r-1}}f\left(\frac{x}{t+1}\right)_{t=0}+\int_{0}^{\infty}dtf\left(\frac{x}{t+1}\right)=H(\ln x)\approx\sum_{n=1}^{\infty}f\left(\frac{x}{n}\right) \qquad (16)$$

H(x) is the Heaviside step function with Mellin transform $\int_0^\infty \frac{dx}{x^{s+1}} H(\ln x) = \frac{1}{s}$, taking the Mellin transform to both side on (16), since $\int_0^\infty \frac{dx}{x^{s+1}} . \sum_{n=1}^{\infty} f\left(\frac{x}{n}\right) = \zeta(s)F(s)$ we reach to the conclusion that the singular solution to (15) is just $y(x) \approx M(x)$ Mertens function , if we put $\left[\sqrt{x}\right]$ instead H(lnx) following the same steps we would find that the singular solution would be $y(x) \approx A(x) = \sum_{n \le x} \lambda(n)$ Liouville function

the fact of a distribution solving a linear ODE should not be surprising ,for example the distribution $g(x) = \delta''(x)$ solves the differential equation (singular solution) $x(1-x)g''(x) + (4-6x)g'(x) - 6g(x) = 0$ for every real value of 'x'

If we differentiate respect to 'x' inside the formula (16) it becomes

$$\frac{\partial}{\partial x}\left(\frac{y(x)}{2x}\right) - \sum_{r=1}^{N} \frac{B_{2r}}{(2r)!} \frac{\partial}{\partial x}\left(\frac{\partial^{2r-1}}{\partial t^{2r-1}} f\left(\frac{x}{t+1}\right).\frac{1}{x}\right)_{t=0} + \frac{y(x)}{x^2} = \frac{\partial}{\partial x}\left(\frac{J(x)}{x}\right) \qquad (17)$$

Y(x)	M(x)	$A(x) = \sum_{n \le x} \lambda(n)$
J(x)	H(lnx)	$\left[\sqrt{x}\right]$

In all cases the derivatives respect to 't' are evaluated at the point t=0 , and J(x) is just the 'source' of the differential equation (16) , depending on the value of J(x) we find different solutions ,for other cases of Dirichlet series of the form $\frac{H(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where the Mellin inverse $\frac{H(s)}{s}$ exists and can be easily calculated, then the source J(x) inside (17) is $2i\pi J(x) = \int_{c-i\infty}^{c+i\infty} ds H(s) x^s s^{-1}$, and the solution to (16) is just $y(x) \approx \sum_{n \le x} a_n = B(x)$, another example is the Chebyshev function that can be obtained by setting

$$y(x) = \sum_{n \le x} \Lambda(n) = \Psi(x) \qquad H(s) = -\frac{d\zeta(s)}{ds} \qquad J(x) = \int_0^x d\left[u\right] \ln(u) \qquad (18)$$

Unfortunately, equations (15) and (1) are only equal in the limit $N \to \infty$, so the solution to (16) will depend on the 'N' used $y(x) = y_N(x)$ in the limit N tending to infinity we will recover the Mertens and Chebyshev functions ,using a suitable source J(x) expressed in mathematical language we have the limits , see Ref [7]

$$\lim_{N\to\infty} y_N(x) = M(x) \qquad \lim_{N\to\infty} y_N(x) = \Psi(x) \qquad J(x) = H(\ln x)\ , \int_0^x d\left[u\right]\ln(u) \tag{19}$$

[x] is the floor function and $\frac{d}{dx}[x] = \sum_{n=0}^{\infty}\delta(x-n)$, depending on the value of ‘N’ the approximation (17) can be worse or better for example if N=2 we have the approximation

$$\sum_{r=1}^{2}\frac{B_{2r}}{(2r)!}\frac{\partial^{2r-1}}{\partial t^{2r-1}}y\left(\frac{x}{t+1}\right)_{t=0} = -x\frac{B_2}{2}y'(x) + \frac{B_4}{24}\left(y''(x)x^2 + 2xy'(x)\right) \tag{20}$$

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