A NEW APPROACH TO RENORMALIZATION USING ZETA REGULARIZATION TECHNIQUES

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Abstract:

In this paper we present a method to deal with divergences in perturbation theory using the method of the Zeta regularization, first of all we use the Euler-MacLaurin Sum formula to associate the divergent integral to a divergent sum in the form $1+2^m+3^m+4^m+\ldots\ldots$ After that we find a recurrence formula for the integrals and apply Zeta regularization techniques to obtain finite results for the divergent series. (Through all the paper we use the notation "m" for the power of the modulus of p, so we must not confuse it with the value of the mass of the quantum particle)

• *Keywords*: -Zeta regularization, Euler sum formula, regulator, Abel-Plana formula

Zeta regularization and recurrence formula:

The Zeta regularization technique is useful when dealing with divergent series, we can use it to associate a finite value to a divergent series in the form:

$$1+2^a+3^a+\ldots \quad a>0 \quad (1) \quad a\in \mathrm{R}$$

Now we could rewrite the series as:

$$1+2^{a-s}+3^{a-s}+\ldots=1+2^{-(s-a)}+3^{-(s-a)}+\ldots=\zeta(s-a) \quad (2)$$

So we can apply the technique of analytic continuation for the Riemann Zeta function to extend it to negative values of the argument s, now if we set s=0 in (2) we could obtain a finite value for our first series as the value , ζ(-a) a>0 , this value can be calculated by means of the contour integral for a given path σ, on the complex plane:

$$\zeta(-a)=\frac{\Gamma(a+1)}{2\pi i}\int_\sigma dq\,\frac{q^{-(a+1)}}{e^{-q}-1} \quad (3)$$

This form of re-calculate a series has been widely used in Physics, one of the most famous examples being the calculation of the force due to "Casimir Effect" using the Zeta regularized value ζ(-3)=-1/120, or as another example, the value ζ(-1)=-1/12 for the divergent sum 1+2+3+4+5+6+7+…. That appears in theoretical physics.

Euler himself, using the operator D=d/dx and Taylor series expansion got the identity for an infinite series with general term f(x+n):

$$f(x)+f(x+1)+f(x+2)+.....=\sum_{n=0}^{\infty}B_n\frac{D^nF}{n!}\quad,\ \frac{dF}{dx}=f(x)\quad(4)$$

If we put f(x)=x or f(x)=x^3 inside (4) and take the limit x→0, we get the finite values –1/12 and –1/120

Now if we introduce the Euler-MacLaurin sum formula in the form:

$$\int_0^b f(x)dx=\frac{f(b)}{2}+\sum_{n=1}^{b-1}f(n)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(b)\quad(5)$$

Where for simplicity we have supposed that f(0)=$f^{(k)}$(0)=0, the B_{2n} that appear in the formula are called the "Bernoulli numbers" ,introduced by us before in formula (4) ,they can be defined in several ways:

$$B_{2n}=-2n\zeta(1-2n)\qquad\qquad \frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}$$

Where the series on the right has convergence radius for |x|<2π. The divergent integrals that appear when doing perturbation theory in Quantum Field theory take the form: (except the case of a Logarithmic-divergent integral m=-1)

$$\int_0^{\Lambda}p^m dp\qquad m\in\mathbf{Z}\qquad\qquad m>0$$

with "p" the modulus of the momentum vector of the particle, now we introduce the regulator Λ→∞, using this and the Euler sum formula we can write the divergent integral in the next form with f(p)=p^m :

$$\int_0^{\Lambda}p^m dp=\frac{\Lambda^m}{2}+\sum_{n=1}^{\Lambda}n^m-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}a_{mr}\Lambda^{m-2r+1}\quad(6)\qquad a_{mr}=\frac{\Gamma(m+1)}{\Gamma(m-2r+2)}$$

$\Gamma(n+1)=n\Gamma(n)=n!$ n 0 or positive integer $\qquad\Gamma(x)=\int_0^{\infty}dxt^{x-1}e^t$ x>0

Now we can rewrite the powers of the regulator in the form of divergent integrals:

$$\Lambda^b=b\int_0^{\Lambda}dpp^{b-1}$$

With the regulator Λ→∞ , the first sum on the left of (5) can be assigned a finite value by Zeta regularization in the form:

$$\sum_{n=1}^{\infty}n^m=1+2^m+3^m+...+\Lambda^m\to\zeta(-m)\quad(7)$$

From the definition of the a_{mr} we can see that for m and r positive integers satisfying m-2r+2<0 the "a" coefficients are 0 so the last series in (7) will be finite, now if we call:

$$I(m,\Lambda)=\int_0^{\Lambda} p^m dp$$

and use the result obtained in (7) we can write the Euler sum formula (5) for our case as:

$$I(m,\Lambda)=(m/2)I(m-1,\Lambda)+\zeta(-m)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}a_{mr}(m-2r+1)I(m-2r,\Lambda) \quad (8)$$

m=1,2,3,……

So to re-calculate our infinite integral I(m,Λ) we should know the values of I(m-1,Λ),I(m-2,Λ),I(m-3,Λ),……I(0,Λ), this last integral value is given by the equality (Using Zeta regularization again) :

$$I(0,\Lambda)=\int_0^{\Lambda} dp=1+1+1+1+.....=\Lambda\rightarrow\zeta(0)$$

The first 4 terms (m=0,1,2,3,4) of the recurrence involving the quantities

$\{a_{nr},\zeta(-n),I(0,\Lambda),B_{2n}\}$ so we could express every UV divergence involving finite quantities plus the "lowest" UV divergence $\int_0^{\infty} dx=1+1+1+1+1+.....$

$$I(0,\Lambda)=I(0,\Lambda)$$

$$I(1,\Lambda)=\frac{I(0,\Lambda)}{2}+\zeta(-1)$$

$$I(2,\Lambda)=\left(I(0,\Lambda)\frac{1}{2}+\zeta(-1)\right)-\frac{B_2}{2}a_{21}I(0,\Lambda)$$

$$I(3,\Lambda)=\frac{3}{2}\left(\frac{1}{2}\left(I(0,\Lambda)+\zeta(-1)\right)-\frac{B_2}{2}a_{21}I(0,\Lambda)\right)+\zeta(-3)-B_2a_{31}I(0,\Lambda)$$

$$I(4,\Lambda)=3\left(\frac{1}{2}\left(I(0,\Lambda)+\zeta(-1)\right)-\frac{B_2}{2}a_{21}I(0,\Lambda)\right)+2\zeta(-3)-2B_2a_{31}I(0,\Lambda)-$$
$$\frac{B_2}{2}\left(\frac{1}{2}\left(I(0,\Lambda)+\zeta(-1)\right)-\frac{B_2}{2}a_{21}I(0,\Lambda)\right)3a_{41}-\frac{B_4}{24}a_{42}I(0,\Lambda)$$

The formula in (7) is a recurrence relation to re-calculate the divergent integrals by using Zeta regularization, we still have to study the case where m is any real number or when m=-1 in that cases by making a change of variable $p=e^{\omega u}$ with w a real number:

$$I(m,\Lambda)=\omega\int_{-\Delta}^{\Delta} du e^{u\omega(m+1)}=\sum_{n=0}^{\infty}\frac{c(n)}{(2n)!}\int_0^{\Delta} du u^{2n} \qquad c(n)=2\omega[(m+1)\omega]^{2n}$$

$\Delta = \frac{\ln \Lambda}{\omega} \to \infty$ for m=-1 we have $I(-1,\Lambda) = 2\omega \int_0^{\Delta} du$ (ultraviolet divergence for the new regulator Δ) $1/\Lambda = \frac{\omega}{\ln \Delta} \to 0$

The "Delta" is the Logarithm of the regulator tending also to infinite, m and w can be arbitrary real numbers, due to the fact that u^{2n+1} are Odd functions then:

$$\int_{-\Delta}^{\Delta} du u^{2n+1} = 0 \quad \text{, n= positive integer}$$

Also the logarithmic divergence can be handled in the form:

$$\ln \Lambda = \zeta(1) - \gamma + O(\Lambda^{-1}) \quad , \sum_{n=1}^{\infty} \frac{1}{n} = \zeta(1)$$

The last is the Euler-Mascheroni constant (γ=0.5772156649015…) ,for s=1 the Riemann zeta function has a pole , taking the Riemann functional equation:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)Cos(\frac{\pi s}{2})\zeta(s)$$

And knowing that for s=(2n+1)π/2, the Cosine is zero with n integer, then the we have that the value ζ(-2n)=0 for n=1,2,3,4,5,6,……. These are the Zeta function trivial zeros.

Abel-Plana formula,Ramanujan resummation and Renormalization:

In mathematics a useful formula involved in calculations for Casimir effect is given by:

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} dx f(x) = \frac{f(0)}{2} + i\int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt \quad (9) \qquad \frac{f(it) - f(-it)}{i} = 2\Im m[f(it)]$$

The derivation of the formula comes from the argument principle for complex (Meromorphic) functions

$$\frac{1}{2\pi i}\int_{\gamma} dz f(z)\frac{g'(z)}{g(z)} = \sum_{\mu} f(x_{\mu}) - \sum_{\tau} f(x_{\tau}) \quad \text{With} \quad g(z) = -if(z)Cot(\pi z)$$

Where the series is made over all the zeros and poles included in closed curve γ, an appropriate election for the contour gives formula (9) and $\lim_{t\to 0}\frac{f(it) - f(-it)}{t} = a$ |a|<∞

The conditions that f(z) must satisfy ($z = x + iy$) are:

$$\int_{a+i\infty}^{b+i\infty} dz \,|\, f(z) \pm g(z)| = 0 \quad , \quad \lim_{y\to\infty} e^{2\pi|y|} \,|\, f(x+iy)| = 0 \text{ for a finite or } \infty \text{ interval of x}$$

Abel-Plana formula allows you to get finite results for divergent integrals in the form :

$$\zeta(-m,\beta)-i\int_0^\infty \frac{(it+\beta)^m-(-it+\beta)^m}{e^{2\pi t}-1}dt-\frac{\beta^2}{2}=\int_o^\infty dp(p+\beta)^m \qquad m\in\square$$

$\zeta(s,\beta)=\sum_{n=0}^{\infty}(n+\beta)^{-s}$ $\Re$e(s)>1 β is a real number different from a negative integer –n.

Hurwitz Zeta function including the Riemann zeta function case β=0 can be extended to negative values, for Negative integer value of argument "s", the values are given by

$\zeta(-m,\beta)=-\frac{B_{m+1}(\beta)}{m+1}$ with $B_m(x)$ the "Bernoulli Polynomial " of grade m.

The Bernoulli Polynomials have the generating function: $\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}$

For the general case or integers μ≥ν≥1 the Functional equation becomes:

$$\zeta(1-s,\frac{\nu}{\mu})=\frac{2\Gamma(s)}{(2\pi\mu)^s}\sum_{r=1}^{\mu}Cos(\frac{\pi s}{2}-\frac{2\pi r\nu}{\mu})\zeta(s,\frac{r}{\mu})$$

For some cases the identity below could be useful:

$\int_0^\infty dt\frac{t^m}{e^{2\pi t}-1}=\frac{\Gamma(m+2)\zeta(m+1)}{(2\pi)^{m+1}}$ m>0

For a logarithmic-type divergence with b>0 integer or real we find:

$\sum_{n\geq 1}^{[R]}(n+b)^{-1}-2\int_0^\infty dt\frac{t}{(t^2+b^2)(e^{2\pi t}-1)}+\frac{1}{2b}=\int_0^\infty\frac{dp}{p+b}$ R=regularized by Ramanujan

resummation , that can be defined recursively by the formula:

$\sum_{n\geq 1}^{[R]}a(n)=a(1)+a(2)+a(3)+.....+a(N-1)+\sum_{n\geq 0}^{[R]}a(n+N)-\int_1^N dta(t)$ N>0, integer.

putting a(x)=1/x we have $\sum_{n\geq 1}^{[R]}\frac{1}{n}\to\gamma$ as N→∞ with

So we could get a finite value for the integrals $\int_0^\infty dp(p+\beta)^r$ using Abel-Plana formula and the values $\zeta(-r,\beta)$ for r∈ R>0, for other cases we should make a change of variables to convert the integral with divergence at p=0 into another integral which diverges for p→∞ with the zeta-regularized value

$$\gamma = \lim_{N\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....+\frac{1}{N}-\ln N)$$

Another form to express the divergent integrals is by means of "distributions" taking the identities:

$$\int_{-\infty}^{\infty} dte^{mt} = \int_{-\infty}^{\infty} dte^{i(-im)t} = 2\pi\delta(-im) = 2\pi\delta(im) = \int_0^{\infty} dxx^{(m-1)} \quad m \geq 1 \qquad \text{t=ln(x)}$$

Using the identity : $\lim_{n\to\infty}\frac{e^{nx}-e^{-nx}}{2i\pi x} = \delta(-ix)$

Using Numerical methods, approximating the integral and its "area" by a sum of rectangles of width ε then we could approximate the integral by :

$$\int_0^{\infty} dx(x+\beta)^m \approx \sum_{n=0}^{\infty}(\beta+\varepsilon n)^m\varepsilon \to \zeta\left(\frac{\beta}{\varepsilon},-m\right)\varepsilon^{m+1} \quad m>0$$

Conclusions:

Using the expressions (8), (6) and the negative values of the Riemann zeta function we could assign a finite value to all the divergent integrals in our calculations to obtain finite results when calculating masses charges or other values in perturbation theory, now using the formula (8) we could reduce every divergent integral to a integral of the form I(m) with m=0, so we would have two kind of divergent integrals I(m,Λ) with m=0,-1 and we could adjust these to measured quantities such as mass or charge as done in the usual renormalization process.

This method can also handle with Non-Renormalizable theories, as it can reduce every divergence to the cases of divergent integrals I(m,Λ), with m=0,-1 by using the recurrence formulae (8) and (6) then, you can apply usual renormalization process to absorb these divergences, This method can also be considered this to give a perturbative approach to Quantum gravity (for low energies, where the Einstein equations are valid and with the metric element, $g_{ab}=n_{ab}+\varepsilon h_{ab}$ $\varepsilon<<1$,n_{ab} is the Euclidean metric), for the case that m is a negative integer m<0, we have taken the change of variable q=1/p so the divergence becomes an ultraviolet divergence (m>0):

$$\int_0^{\Lambda} dpp^{-m} \to \int_0^{\Lambda} dqq^{m-2} \quad m \neq 1 \text{ ,m>0}$$

As an example of our method, let be a theory with its Lagrangian $L(a,b,\phi,\partial_\mu\phi)=L_0+gV_{int}$ with g a (finite, usually $g=g_R$) coupling constant, and (a,b) some "bare" quantities of our theory (mass, charge….) , then using the formulae (6) and (8), we can reduce every divergence to the cases m=0,-1 recursively by introducing the values ζ(1-2n) n=1,2,3,….., so for our theory we have only 2 divergent quantities left, getting:

$$\sum_n c_n g^n I(m,\Lambda) \to \sum_n d_n g^n \zeta(1-2n) + A + B$$

Where A and B are divergent quantities associated to integrals of the form, I(0, Λ) and I(-1, Λ) now, we can use the usual renormalization process to get rid of these 2

divergent quantities A and B by measuring the real values of the "bare" quantities a_R , b_R experimentally, so in the end we put : (renormalized=$_R$)

$(a_R , b_R) \leftarrow (A,B)=(A(\Lambda),B(\Lambda))$ when $\Lambda\to\infty$.

Our algorithm of renormalization works in this form, first you express any divergent integral I(m, Λ) m=1,2,3,4…. using the recursive relation (8) as a linear combination of ζ(1-2n) regularized values and the I(0,Λ) divergent integral ,reducing every divergence to the cases I(0,Λ) and I(-1,Λ), next you use the usual renormalization process to get rid of these 2 divergent integrals.

Although we have used a recursive formula to obtain only 2 divergent integrals I(0, Λ) and I(-1, Λ), we could use "Zeta regularization" again and the technique called "Ramanujan Resummation" (see ref [2]) to obtain finite values for these 2 integrals , in particular:

$$\zeta(0)\to -B_1 \qquad \text{and} \qquad \sum_{n\geq 1}^{[R]}\frac{1}{n}\to\gamma$$

Where "R" stands for Ramanujan Resummation of the Harmonic series, and B_1=1/2 is the first Bernoulli number, the former is the "finite" value for the "lowest" UV divergence, the latter is the associated finite value used for the logarithmic divergence, related to the Harmonic series by the identity:

$$\gamma=\lim_{\Lambda\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....+\frac{1}{\Lambda}-\ln\Lambda) \text{ and } \lim_{\Lambda\to\infty}\frac{1+1/2+1/3+...+1/\Lambda}{\ln\Lambda}=1$$

Although Hardy in his Book "*divergent series*" considered the case: (page 40 at 2.3.5)

$$\eta(1-2k)=1-2^{2k-1}+3^{2k-1}-.....=(-1)^{k-1}\frac{2^{2k}-1}{2k}B_{2k} \quad \text{(Modern notation so } B_{2k+1}=0)$$

As the "sum" of the divergent series, using the identity $\eta(1-2k)=(1-2^{2k})\zeta(1-2k)$ and $\eta(0)=-\zeta(0)=1-1+1-1+1-.....=1/2$, we can find the values of ζ(1-2k) k≥1.

For the sum of the series $\sum_n d_n g^n\zeta(1-2n)$,using Bernoulli numbers we could write $-\sum_n g^n d_n\frac{B_{2n}}{2n}$ if we had the Power series generating function F(x) so it satisfied:

$\sum_{n=0}^{\infty}d_n\frac{B_{2n}}{2n}\frac{x^n}{n!}=F(x)$ and $n!=\int_0^{\infty}dxe^{-x}x^n$,Then we could write its "Borel sum"B(S) as the integral

$\int_0^{\infty}dxe^{-x}F(x)=B(S)$ so taking "n" big the sum could be approximated by the integral:

$\int_0^{\infty}dxe^{-x}F(xg)=B_1(S)\approx-\sum_{n=0}^{N}g^n d_n\zeta(1-2n)$ N $\to\infty$ with $F(x)=O(e^{Mx})$ M>0, So the "Borel transform" makes sense and the "sum" converges to a finite value, the Borel sum is a mathematical summation method to assign a finite value to the divergent series:

$|B_1(S)| \rightarrow |\sum_n g^n d_n \zeta(1-2n)|$ S stands for "Sum".

Using the nomenclature of Z-transform with z=1/x, G(z)=F(1/z) we would have a formula for the coefficients in terms of a Contour integral and Bernoulli numbers:

$$\frac{(n+1)!i\pi}{B_{2n}g^{n}}\int_C dzG(z)z^{n-1} = d_n$$,C= closed path including the origin and the poles of G.

Other methods we have studied to deal with divergences are the Abel-Plana formula (10) to give a meaning to divergent integrals with a real r>0 and Ramanujan-resummation for the case of logarithmic divergent integrals.

The main task of this paper has been to provide a "finite" value for divergent integrals that yield to UV divergences, either using a recursion formula (6) and (8) via MacLaurin sum formula or use the relationship between an infinite series and its integral by Abel-Plana formula, and apply Zeta regularization to "absorb" the divergent series $1+2^m+3^m+\ldots\ldots\ldots\ldots$

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