AN EULER-MC LAURIN FORMULA FOR INFINITE DIMENSIONAL SPACES

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ABSTRACT:=In this paper we study a infinite dimensional generalization of Euler-Mc Laurin sum formula to deal with Functional integration, our main task is to produce a Remainder formula for an analogue of Trapezoidal rule on infinite dimensional spaces using the sum involving $\sum_{m=-\infty}^{\infty} F[\phi_{classical} + \varepsilon\delta(x-y)]$ to be able to calculate integrals involving functionals, with many applications to Mathematical Physics.

- *Keywords:* = Euler-Mc Laurin sum formula, Functional integral and derivative

1. INTRODUCTION

One of the open problems in Mathematical Physics, is to define Infinite dimensional integrals over functional spaces, given in the form:

$$\int D[\phi]\exp(-aS[\phi]) \qquad \text{With} \qquad S[\phi] = \int d^4xL(\phi, \partial_\mu \phi) \qquad (1.1)$$

Where a is either a positive parameter (Wigner Mathematical formulation of Brownian Motion), or a Pure imaginary Number (Feynmann formulation of Quantum Mechanics). One of the forms to view these integrals is to discretize the infinite dimensional space, and calculate the Integral:

$$\int dx_1 \int dx_2 .... \int dx_N F(x_1, x_2, x_3, ..., x_n) \to \int D[\phi]F[\phi] \qquad n \to \infty \quad (1.2)$$

Unfortunately to define the Volume for an infinite dimensional space, we find many problems mainly, the Volume of an n-dimensional Hypercube of side 'a':

$$V(u) = \begin{cases} 1 & \text{iff} \quad \text{u=1} \\ 0 & \text{iff} \quad \text{u<1} \\ \infty & \text{iff} \quad \text{u>1} \end{cases} \qquad \text{for} \quad n \to \infty \qquad (1.3)$$

However if the parameter 'a' is big we could use a Saddle-point expansion near the extremal of $S[\phi]$, having the expansion:

$$S[\phi] \approx S[\phi_{classical}] + \frac{1}{2}\int d^4x_1 \int d^4x_2 \frac{\delta^2 F}{\delta\phi_{classical}(x_1)\delta\phi_{classical}(x_2)}(\phi - \phi_{classical}) + ...$$

(1.4)

With $\delta F[\phi_{classical}] = 0$, this is the WKB Semiclassical approach , Its physical meaning is that we are just keeping terms upto $\hbar$, then our Functional integral (1.1) becomes just an Infinite dimensional Gaussian analogue of the expression:

$$\int \prod_i dx_i e^{B_{ij}x_i x_j} \qquad (1.5)$$

For the case of an 'Scalar Field', we find (C is a constant):

$$S[\phi] = \int d^4x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu \partial_\nu \phi - \frac{1}{2}m\phi^2 \right) \qquad \int D[\phi]e^{iS/\hbar} = CDet\left(A\right) \qquad (1.5)$$

A is the Functional determinant $A = -\partial^2 - m^2$

As a brief *'Summary'* before presenting our formula:

- The function $\phi(x)$ is a Vector of a Functional space with Components $\phi(\zeta_i)$ $\zeta_i \in R^4$
- The functional derivative $\frac{\delta}{\delta\phi}$ is the analogue to the Gradient ∇
- The scalar product $\sum_i x_i u_i \to \int dt x(t)u(t)$

2. FUNCTIONAL EULER-MC LAURIN FORMULA

In this paper, we will provide an approximate Numerical method to define the Functional integral, using the series $\sum_{m=-\infty}^{\infty} F[\phi_{classical} + mT\delta(x-y)]$ and an Euler-Mc-Laurin formula to define the 'Remainder'.

Many identities valid for finite-dimensional spaces can be generalized to fit them into an infinite dimensional space:

$$\sum_{i=0}^{n} x_i \nabla_i \to \int d^4x \phi(x)\frac{\delta}{\delta\phi} \quad \text{as n} \to \infty \quad \text{(scalar product)} \quad (2.1)$$

In (2.1) we have introduced the Functional derivative ,Which can be defined in the form:

$$\lim_{\varepsilon\to 0}\frac{F[\phi+\varepsilon\delta(x-y)]-F[\phi]}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{dF(\phi+\varepsilon\delta(x-y))}{d\varepsilon}=\frac{\delta F[\phi]}{\delta\phi(y)} \qquad (2.2)$$

(2.2) is the analogue of the directional derivative (finite dimensional case) in the direction of the vector $\vec{\varepsilon}=(\varepsilon,\varepsilon,\varepsilon,...............)$, also we can give the analogue to the 'Integration by parts' on R^n :

$$\int_\Omega dx\nabla u\cdot\mathbf{v}=-\int_\Omega dxu\nabla\cdot\mathbf{v} \qquad \text{(vanishing Boundary term } \int_{\partial\Omega} dxu\vec{n}\cdot\mathbf{v}=0\text{)} \qquad (2.3)$$

$$\int D[\phi]\frac{\delta F[\phi]}{\delta\phi}G[\phi]=-\int D[\phi]\frac{\delta G[\phi]}{\delta\phi}F[\phi] \qquad (2.4)$$

And the Poisson Sum-formula for Functional integrals:

$$\sum_{m=-\infty}^{\infty}F[\phi_{classical}+mT\delta(x-y)]=\int D[\phi]F[\phi+\phi_{classical}]\left(\sum_{k=-\infty}^{\infty}e^{\frac{2\pi ik}{T}\int d^4x\phi(x)}\right) \qquad (2.5)$$

$$\omega[\phi]=\sum_{k=-\infty}^{\infty}e^{\frac{2\pi ik}{T}\int d^4x\phi(x)}=\omega[\phi+T\delta(x-y)] \qquad (2.6)$$

To prove (2.5) we can use simply Fourier Analysis and the change of variables:

$$\phi\to\phi'=\phi+m\delta(x-y) \qquad D[\phi]\to D[\phi']=D[\phi] \qquad (2.7)$$

So our measure must be traslational invariant, for any measure which is not traslational invariant we could write in the form:

$D[\phi]M[\phi]$ And the classical action becomes $S[\phi]\to S[\phi]+\left(\frac{\hbar}{i}\right)\ln M[\phi]$ (2.8)

Taking $T\to 0$ inside (2.5) , only the term with k=0 is relevant so we have:

$$\lim_{T\to 0}\left(\sum_{m=-\infty}^{\infty}F[\phi_{classical}+mT\delta(x-y)]\right)=\int D[\phi]F[\phi+\phi_{classical}] \qquad (2.9)$$

Then it seems that our 'ansatz' to approximate the Functional integral by a sum $\sum_m F[\phi_{classical}+m\varepsilon\delta(x-y)]$ with step 'ε' seems accurate if we make this step small (as an analogy to the Trapezoidal rule on R^n), for $F[\phi]=\exp(iS[\phi]/\hbar)$

$$\lim_{\hbar\to 0}\left(\sum_{m=-\infty}^{\infty}\exp(iS[\phi_{classical}+mT\delta(x-y)]/\hbar\right)\approx e^{iS[\phi_{classical}]/\hbar} \qquad (2.10)$$

Which is just the classical result for the Feynmann integral, to improve this we would need to find some kind of 'Remainder' expression, using a generalization of the Euler-Mc Laurin sum formula, involving functional derivatives.

The condition (2.4) applied to QFT taking the Functional integral , setting $\int D[\phi]e^{\frac{i}{\hbar}S[\phi]}=1$ (renormalization condition) implies:

$$\int D[\phi]e^{\frac{i}{\hbar}S[\phi]}\frac{\delta F[\phi]}{\delta\phi}=-\frac{i}{\hbar}\int D[\phi]e^{\frac{i}{\hbar}S[\phi]}F[\phi]\frac{\delta S[\phi]}{\delta\phi} \qquad (2.11)$$

$$\left\langle\Psi\right|\frac{\delta F[\phi]}{\delta\phi}\left|\Psi\right\rangle=\frac{-i}{\hbar}\left\langle\Psi\right|F[\phi]\frac{\delta S[\phi]}{\delta\phi}\left|\Psi\right\rangle$$

(Where Ψ is a given state of a QFT theory with Classical Action 'S')

On condition F is a Polynomially Bounded Functional.

Using the 'Generating Functional' for QFT Z[J] we can write (2.5) as

$$Z[J]=\int D[\phi]e^{\frac{i}{\hbar}\left(S[\phi]+\int d^4xJ(x)\phi(x)\right)} \qquad \sum_{m=-\infty}^{\infty}F[\phi_{classical}+mT\delta(x-y)]=\sum_{k=-\infty}^{\infty}Z[J=\frac{2k\hbar}{T}]$$

(2.12)

To define the Euler-Mc Laurin analogue for Infinite-dimensional spaces, first we Introduce the 'Traslational operator' :

$$e^{D_\infty}=\exp\left(\int d^4x\frac{\delta}{\delta\phi}\right) \qquad e^{\varepsilon D_\infty}F[\phi]=F[\phi+\varepsilon\delta(x-y)] \qquad (2.13)$$

For $\varepsilon\to 0$, we must recover the Functional derivative:

$$\lim_{\varepsilon\to 0}\left(e^{\varepsilon D_\infty}-1\right)F[\phi]=\left(\int d^4x\frac{\delta F[\phi]}{\delta\phi}\delta(x-y)\right)=\frac{\delta F}{\delta\phi(y)} \qquad (2.14)$$

Also using the expansion (involving Bernoulli Numbers B_n) of the functional:

$$\frac{\int d^4x\frac{\delta}{\delta\phi}}{e^{\int d^4x\frac{\delta}{\delta\phi}}-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}\int d^4x_1......\int d^4x_n\frac{\delta^n}{\delta\phi(x_1)\delta\phi(x_2).........\delta\phi(x_{2n})} \qquad (2.15)$$

Hence, we can define the generalization of Euler-Mc Laurin formula for infinite dimensional spaces in the form:

$$\int_{\phi_0} D[\phi]F[\phi] = \frac{F[\phi_0]}{2}\varepsilon + \varepsilon\sum_{m=1}^{\infty} F[\phi_0 + m\varepsilon\delta(x-y)] + R \qquad (2.16)$$

$$\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\varepsilon^{2r}\int d^4x_1......\int d^4x_{2r-1}\,\frac{\delta^{2r-1}F[\phi]}{\delta\phi(x_1)\delta\phi(x_2).........\delta\phi(x_{2r-1})} = R \qquad (2.17)$$

Where all the functionals derivatives $\delta^{2r-1}F[\phi]$ are evaluated at $\phi = \phi_0$, this formula with the Remainder gives an (approximate) numerical value to the Functional integral using the series $\varepsilon\sum_{m=1}^{\infty} F[\phi_0 + m\varepsilon\delta(x-y)]$, as $\varepsilon \to 0$ the Remainder becomes less significant, we think this formula can be useful to define Functional integration without recalling to discretization of the space of Paths or Montecarlo integration, however for many cases the factor:

$$\frac{\delta^{2r-1}F[\phi]}{\delta\phi(x_1)\delta\phi(x_2).........\delta\phi(x_{2r-1})} \qquad (2.18)$$

exists only in the sense of 'distribution theory' and needs to be regularized.

Note, that for R^n involving a smooth function f, the Remaineder 'R' : $(\varepsilon = 1)$

$$\frac{1}{2}f(x_0) + \sum_n f(x_0 + n\vec{\eta}) + \frac{(\vec{\nabla}.\vec{\eta})}{e^{(\vec{\nabla}.\vec{\eta})} - 1}f(x_0) = R \qquad \vec{\eta} = (1,1,1,.........) \qquad (2.19)$$

$(\vec{\nabla}.\vec{\eta})f \quad \to$ 'Directional derivative on $\quad R^n \qquad x_0 \in R^n$

Expressions (2.16) and Remainder 'R' (2.17) are the generalization of the formula:

$$\int_x^{\infty} f(t)dt = \varepsilon\frac{f(x)}{2} + \varepsilon\sum_{n=x+\varepsilon}^{\infty} f(n) + \sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\partial_x^{\,2r-1}f(x)\varepsilon^{2r} \qquad (2.20)$$

$f(x) \to 0$ as $x \to \infty$

And we have the equivalence:

$$\int_{\phi_0} D[\phi]F[\phi] \to \int_{a_1}^{\infty} dx_1\int_{a_2}^{\infty} dx_2.......\int_{a_n}^{\infty} dx_n F(x_1, x_2, ..., x_n) \quad n \to \infty \quad (2.21)$$

Where the function (vector) ϕ_0 has components $\phi_0(\xi_n) = a_n$

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