

Concerning radii in Einstein’s gravitational field

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ABSTRACT

It is proved herein that the quantity ‘ r ’ appearing in the so-called “Schwarzschild solution” is neither a distance nor a geodesic radius but is in fact the radius of Gaussian curvature. The radius of Gaussian curvature does not determine the geodesic radial distance (the proper radius) from the arbitrary point at the centre of spherical symmetry of the Schwarzschild manifold. It does not directly determine any distance at all in the Schwarzschild manifold.

Key words: – black hole physics, relativity, gravitation.

1 INTRODUCTION

In the usual interpretation of Hilbert’s “Schwarzschild’s solution” (Abrams 1989; Antoci 2001; Longe 2002), the quantity r therein has *never* been properly identified. It has been variously and vaguely called “the radius” of a sphere (Mould 1994; Dodson & Poston 1991), the “coordinate radius” (Wald 1994), the “radial coordinate” (Carroll & Ostle 1996; Misner et al. 1973), the “radial space coordinate” (Zel’dovich & Novikov 1996), the “radius of a 2-sphere” (Bruhn 2008), the “areal radius” (Wald 1994; Ludvigsen 1999), the “reduced circumference” (Taylor & Wheeler 2000), and even a “*a gauge choice: it defines the coordinate r*” (’t Hooft 2008). In the particular case of $r = 2GM/c^2$ it is invariably referred to as the “Schwarzschild radius” or the “gravitational radius”. However, the irrefutable geometrical fact is that r , in Hilbert’s version of the Schwarzschild/Droste line-element, is the radius of Gaussian curvature (Levi-Civita 1977; Crothers 2007; Schwarzschild 1916a; Crothers 2005), and as such it *does not* in fact determine the geodesic radial distance from the centre of spherical symmetry located at an arbitrary point in the related metric manifold. Indeed, it does not in fact determine any distance at all in the Schwarzschild manifold. It is the radius of Gaussian curvature merely by virtue of its formal geometric relationship to the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section.

It must also be emphasized that a geometry is completely determined by the *form* of its line-element (Tolman 1987).

2 GAUSSIAN CURVATURE

Recall that Hilbert’s version of the “Schwarzschild” solution is (using $c = G = 1$),

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1)$$

wherein r can, by assumption (i.e. without any proof), in some way or another, go down to zero, and m is allegedly the mass of the source of the gravitational field. Schwarzschild’s actual solution (Schwarzschild 1916b), for comparison, is

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \\ R = R(r) = (r^3 + \alpha^3)^{\frac{1}{3}}, \quad 0 \leq r < \infty, \quad \alpha = \text{const.} \quad (2)$$

Note that the metric tensor of (2) is singular only when $r = 0$ (in which case the metric does not actually apply), and that the constant α is indeterminable (Schwarzschild did not assign any value to the constant α for this reason).

For a 2-D spherically symmetric geometric surface (O’Neill 1966) determined by

$$ds^2 = R_c^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad R_c = R_c(r), \quad (3)$$

the Riemannian curvature (which depends upon both position and direction) reduces to the Gaussian curvature K (which depends only on position), given by (Levi-Civita 1977; Pauli 1981; Kay 1988; McConnell 1957; Landau & Lifshitz 1951)

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$$K = \frac{R_{1212}}{g},$$

where $R_{ijkm} = g_{in}R^n{}_{jkm}$ is the Riemann tensor of the first kind and $g = g_{11}g_{22} = g_{\theta\theta}g_{\varphi\varphi}$ (because the metric tensor is diagonal). Recall that

$$R^1{}_{212} = \frac{\partial\Gamma^1{}_{22}}{\partial x^1} - \frac{\partial\Gamma^1{}_{21}}{\partial x^2} + \Gamma^k{}_{22}\Gamma^1{}_{k1} - \Gamma^k{}_{21}\Gamma^1{}_{k2},$$

$$\Gamma^\alpha{}_{\alpha\beta} = \Gamma^\alpha{}_{\beta\alpha} = \frac{\partial}{\partial x^\beta} \left(\frac{1}{2} \ln |g_{\alpha\alpha}| \right),$$

$$\Gamma^\alpha{}_{\beta\beta} = -\frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{\beta\beta}}{\partial x^\alpha}, \quad (\alpha \neq \beta),$$

and all other $\Gamma^\alpha{}_{\beta\gamma}$ vanish. In the above, $k, \alpha, \beta = 1, 2$, $x^1 = \theta$ and $x^2 = \phi$, of course. Straightforward calculation gives for expression (3),

$$K = \frac{1}{R_c^2},$$

so that R_c is the inverse square root of the Gaussian curvature, i.e. the radius of Gaussian curvature, and so r in Hilbert's "Schwarzschild's solution" is the radius of Gaussian curvature. The geodesic (i.e. proper) radius, R_p , of a spatial section of Schwarzschild's solution (2), up to a constant of integration, is given by

$$R_p = \int \frac{dR(r)}{\sqrt{1 - \frac{\alpha}{R(r)}}}, \quad (4)$$

and for Hilbert's "Schwarzschild's solution" (1), by

$$R_p = \int \frac{dr}{\sqrt{1 - \frac{2m}{r}}}.$$

Thus the proper radius and the radius of Gaussian curvature *are not the same*. The radius of Gaussian curvature does not determine the geodesic radial distance from the arbitrary point at the centre of spherical symmetry of the metric manifold. It is a "radius" only in the sense of it being the inverse square root of the Gaussian curvature. A detailed development of the foregoing, from first principles, is given elsewhere (Levi-Civita 1977; Crothers 2007).

Note that in (2), if $\alpha = 0$ Minkowski space is recovered:

$$ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

$$0 \leq r < \infty.$$

In this case the radius of Gaussian curvature is r and the proper radius is

$$R_p = \int_0^r dr = r,$$

so that the radius of Gaussian curvature and the proper radius are identical. It is for this reason that in the space-time of Minkowski the radius of Gaussian curvature can be substituted for the proper radius (i.e. the geodesic radius). However, in the case of a (pseudo-) Riemannian manifold, such as (1) and (2) above, only the great circumference and surface area can be directly determined via the radius of Gaussian curvature. Distances from the arbitrary point at the centre of spherical symmetry to a geodesic spherical surface in a Riemannian metric manifold can only be determined via the proper radius, except for particular points (if

any) in the manifold where the radius of Gaussian curvature and the geodesic radius are identical, and volumes by a triple integral involving a function of the radius of Gaussian curvature. In the case of Schwarzschild's solution (2) (and hence also for (1)), the radius of Gaussian curvature, $R_c = R(r)$, and the proper radius, R_p , are identical only at $R_c \approx 1.467\alpha$. When the radius of Gaussian curvature, R_c , is greater than $\approx 1.467\alpha$, $R_p > R_c$, and when the radius of Gaussian curvature is less than $\approx 1.467\alpha$, $R_p < R_c$.

The upper and lower bounds on the Gaussian curvature (and hence on the radius of Gaussian curvature) are not arbitrary, but are determined by the proper radius in accordance with the intrinsic geometric structure of the line-element (which completely determines the geometry), manifest in the integral (4). Thus, one cannot merely assume that the radius of Gaussian curvature for (1) and (2) can vary from zero to infinity. Indeed, in the case of (2) (and hence also of (1)), as R_p varies from zero to infinity, the Gaussian curvature varies from $1/\alpha^2$ to zero and so the radius of Gaussian curvature correspondingly varies from α to infinity, as easily determined by evaluation of the constant of integration associated with the indefinite integral (4). Moreover, in the same way, it is easily shown that expressions (1) and (2) can be generalised (Crothers 2005) to all real values, but one, of the variable r , so that both (1) and (2) are particular cases of the general radius of Gaussian curvature, given by

$$R_c = R_c(r) = \left(|r - r_0|^n + \alpha^n \right)^{\frac{1}{n}},$$

$$r \in \mathfrak{R}, \quad n \in \mathfrak{R}^+, \quad r \neq r_0,$$

$$(5)$$

wherein r_0 and n are entirely arbitrary real constants. Choosing $n=3$, $r_0=0$ and $r > r_0$ yields the solution (2) actually obtained by Schwarzschild. Choosing $n=1$, $r_0=\alpha$ and $r > r_0$ yields line-element (1) as determined by Johannes Droste (1917) in May 1916, independently of Schwarzschild. Choosing $n=1$, $r_0=\alpha$ and $r < r_0$ gives $R_c = 2\alpha - r$, with line-element

$$ds^2 = \left(1 - \frac{\alpha}{2\alpha - r} \right) dt^2 - \left(1 - \frac{\alpha}{2\alpha - r} \right)^{-1} dr^2 - (2\alpha - r)^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

Using relations (5) directly, all real values of $r \neq r_0$ are permitted. In any case, however, the related line-element is singular only at the arbitrary parametric point $r=r_0$ on the real line (or half real line, as the case may be), which is the only parametric point on the real line (or half real line, as the case may be) at which the line-element fails (at $R_p(r_0)=0 \forall r_0 \forall n$). Indeed, substituting (5) for $R(r)$ in (4), and evaluating the constant of integration gives

$$R_p = \sqrt{R_c(R_c - \alpha)} + \alpha \ln \left(\frac{\sqrt{R_c} + \sqrt{R_c - \alpha}}{\sqrt{\alpha}} \right),$$

where $R_c = R_c(r)$ is given by (5).

Note that in the Standard Model interpretation of (1), only g_{00} and g_{11} are modified by the presence of the constant m . However, according to (2) and (5) *all* the components of the metric tensor are modified by the constant α , and since (1) is a particular case of (5), all the components of the metric tensor of (1) are modified by the constant α as well.

The Kruskal-Szekeres coordinates do not take account of the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section of the Schwarzschild manifold. These coordinates thereby violate the geometric form of the line-element, producing a completely separate pseudo-Riemannian manifold that does not form part of the solution space of the Schwarzschild manifold (Smoller & Temple 1998), and are consequently invalid. The concept of the Black Hole is therefore invalid.

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