Orbital Precession with no Assumptions

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Abstract

The anomalous precession of the planet Mercury's orbit was originally explained by General relativity and subsequently by numerous others all of whom, including Einstein made assumptions. Einstein probably made the most radical assumption by proposing that space was curved. Others have modified Newtonian gravity in various ways. Biswas [2] has shown that a Lorentz covariant modification of the classic Newtonian gravitational potential can yield the correct precession. Others [4] have made more radical assumptions such as modifying the gravitation field equations to parallel the field equations of Electromagnetism [3,8] or assuming a relativistic Lagrangian that looks surprisingly like a Schwarzschild metric [4]. About a year ago, this author published a paper on this site in which assumptions were made regarding methods of combining the equations of SR with Newtonian mechanics. In this paper I will show that N0 (ZERO) assumptions need be made. In essences, orbital precession was already built into physics prior to GR.

Keywords: Planetary Precession, General Relativity, Orbital Precession of mercury, Flat Space

Background physics

The energy of a relativistic particle in motion is given by:

$$E = m_0c^2 + {m_o v^2}\big/{2} + {3m_0v^4}\big/{8c^2} + \ldots\ldots \qquad 1.10$$

These terms are all we will use as additional terms only become relevant at extreme velocities. The kinetic energy of the particle will then be:

$$T = {m_o v^2}\big/{2} + {3m_0v^4}\big/{8c^2} \qquad 1.20$$

From Newtonian mechanics the total velocity of an object moving under the influence of a central force is given by [6]:

$$v^2(r) = \left(L^2/m^2P^2\right)\left[\left(e^2-1\right)+\left(2P/r\right)\right] \qquad 1.30$$

Where $P = L^2 \big/ GMm^2$ and L is the total constant angular momentum and e is the eccentricity of the orbit.

Equations 1.20 and 1.30 are all the physics that is needed. The rest is mathematics.

The Math

To develop the necessary mathematical relationships needed we first explore the relationship between the Lagrangian operators on general power functions.
The Lagrangian operators are:

$$L_t = \frac{d}{dt}\frac{\partial}{\partial v} \quad \text{and} \quad L_x = \frac{\partial}{\partial x}$$

In one dimensional mechanics, given the initial conditions one can determine both the position as a function of the time x(t) and the velocity as a function of time v(t). Similarly time can be written (by taking the inverse function) as a function of position t(x), and hence velocity can be written as a function of position v(x).
Take a general function of the form $a + kv^n(t)$, where n is a positive integer ≥ 2 and a is a constant. The function can be written as:

$$f(t) = a + kv^n(t) \;=\; f(x) = a + kv^n(x) \quad \text{where } x = x(t)$$

Applying L_t to $f(t)$ we find:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(a+kv^n\right) = \frac{d}{dt}\left(nkv^{n-1}\right) = n(n-1)kv^{n-2}\frac{dv}{dt} \qquad 1.40$$

Applying L_x to $f(x)$ we find:

$$\frac{\partial}{\partial x}(kv^n(x)) = nkv^{n-1}\frac{\partial v}{\partial x} = nkv^{n-2}\frac{dx}{dt}\frac{\partial v}{\partial x} = nkv^{n-2}\frac{dv}{dt} \qquad 1.50$$

In 1.50, kv^{n-1} was written as $kv^{n-2}\dfrac{dx}{dt}$

From 1.40 and 1.50 it is clear that:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(kv^n(t)\right) = (n-1)\frac{\partial}{\partial x}\left(kv^n(x)\right) \qquad 1.60$$

To extract the actually Lagrangian form we use the function $T = \frac{1}{2}mv^2$:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(T(t)\right) = \frac{\partial}{\partial x}\left(T(x)\right) \qquad 1.70$$

1.70 is valid regardless of any constants added to T so we can write:

$\frac{d}{dt}\frac{\partial}{\partial v}\left(T(t)\right) = \frac{\partial}{\partial x}\left(-E_T + T(x)\right)$ Where E_T is the total energy which is constant $-E_T + T(x)$ is just –V(x), the potential energy which is assumed to be a function

only of x.

Since L_t operating on V(x) is zero and L_x operating on $T(t)$ is zero we can write 1.70 as:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(T(t) - V(x)\right) = \frac{\partial}{\partial x}\left(T(t) - V(x)\right) \qquad 1.80$$

1.80 is in the form of the classical Lagrangian.

We can use 1.70 to generate the gravitational potential energy function of an orbiting body by noting that 1.30 expresses the total velocity and hence the total kinetic energy as a function of r. The problem has only one independent variable, r and hence we can use the above one dimensional reasoning.

Using:

$T(t) = \frac{1}{2}mv^2(t)$ and $T(r) = \frac{1}{2}mv^2(r)$ and substituting 1.30 for $v^2(r)$ then differentiating we get:

$ma = -{}^{GMm}\!/_{r}$, which is the classic Newtonian formula.

We just showed that, knowing the velocity as a function of r 1.70 allows one to generate the potential energy function.

We can now proceed to solve the problem of orbital precession.

Using 1.20 and 1.60 we get the equalities:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{1}{2}mv^2\right)= \frac{\partial}{\partial r}\left(\frac{1}{2}mv^2(r)\right) \qquad \text{and} \qquad \frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{3mv^4}{8c^2}\right)= 3\frac{\partial}{\partial r}\left(\frac{3mv^4(r)}{8c^2}\right)$$

Combining terms we get the equality:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{1}{2}mv^2\right)+\frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{3mv^4}{8c^2}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}mv^2(r)\right)+3\frac{\partial}{\partial r}\left(\frac{3mv^4(r)}{8c^2}\right) \qquad 1.90$$

We don't know v(r) exactly but for most problems such as the orbit of the Planet Mercury we know that it very closely approximates the classical v(r). We will then use 1.30 on the right side. Squaring 1.30 to get $v^4(r)$ and differentiating we get.

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{1}{2}mv^2\right)+\frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{3mv^4}{8c^2}\right) = -\frac{GMm}{r^2}-\frac{3mk_1}{2r^2}-9\frac{G^2M^2m}{r^3c^2} \qquad 2.00$$

$$k_1 = \frac{3G^3M^3m^2(e^2-1)}{L^2c^2}.$$

k_1 is very small compared with the first and has no effect on the orbital precession but has some interesting properties that we shall discuss later.

What we want to solve 2.00 for $r=\int vdt$. The velocity v however appears in both terms on the left. We would like to get it in a single term.
Define the following kinetic energy functions:

$$T_2(v)=\frac{3mv^4}{8c^2} \qquad T_1(v) = \frac{mv^2}{2} \quad \text{and} \quad T_T(v)=T_1(v)+T_2(v)$$

2.00 becomes:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(T_T(v)\right)=-\frac{GMm}{r^2}-\frac{3mk_1}{2r^2}-9\frac{G^2M^2m}{r^3c^2} \qquad 2.10$$

Next we define a potential function as follows:

$$\phi(r)=-\frac{GM}{r}-\frac{3k_1}{2r}-\frac{9}{2}\frac{G^2M^2}{r^2c^2}$$

To get this in the form of 1.60 we write:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(T_T(v)\right)=(n-1)\frac{\partial}{\partial r}\left(T_T(r)\right) \quad \text{where n=2 and } T_T(r)=-m\phi(r)$$

Next we note that:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(T_T(v)-T_2(v)\right)=\frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{mv^2}{2}\right)=(n-1)\frac{\partial}{\partial r}\left(T_T(r)-T_2(r)\right) \quad \text{where n=2}$$

$$\frac{\partial}{\partial r}\left(T_T(r)\right)=-\frac{GMm}{r^2}-\frac{3mk_1}{2r^2}-9\frac{G^2M^2m}{r^3c^2}$$

$$\frac{\partial}{\partial r}\left(T_2(r)\right)=-\frac{mk_1}{2c^2r^2}-3\frac{G^2M^2m}{r^3c^2}$$

Combining term gives:

$$\frac{d}{dt}\frac{\partial}{\partial v}\left(\frac{1}{2}mv^2\right)=-\frac{GMm}{r^2}-\frac{mk_1}{r^2}-6\frac{G^2M^2m}{r^3c^2} \qquad 2.20$$

2.20 can be solved exactly. We could either ignore the second term or combine it with the first effectively skewing the Gravitational constant. Interesting, it is a function of eccentricity and depending on the orbit, it can be either positive negative or zero. For an elliptic orbit, such as that of the Pioneer 10 space probe (e=1.737) it increases the gravitational constant. It may be at least partially responsible for the anomalous Doppler readings.

We can proceed to solve 2.20, ignoring the second term both because it is very small and because it has no effect on the precession.

In polar coordinates 2.20 results in two equations, one from the r coordinate and one from the θ coordinate

The details of solving this equation for $r(\theta)$ can be found in any text book and many online sites [6].

In summary the solution of the θ equation is independent of the nature of the potential field and results in conservation of angular momentum designated L. The following conversions are then used on the radial equation:

$$\frac{d}{dt}=\frac{d\theta}{dt}\frac{d}{d\theta}=\frac{L}{mr^2}\frac{d}{d\theta}$$

And $$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d\left(1/r\right)}{d\theta}$$

And $$u \equiv \frac{1}{r}$$

Applying these conversions yields:

$$u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{u^2}{l} + 6\left(\frac{h}{cl}\right)^2 u^3 \qquad 2.50$$

Where $h = L/m$, the angular momentum per unit mass and $l = h^2/GM$. c is of course the speed of light. l is a measurable property of an orbit and is designated the semi-latus rectum. It is related to the eccentricity.

Simplifying 2.50 yields:

$$\frac{d^2u}{d\theta^2} + \left(1-6\left(\frac{h}{cl}\right)^2\right)u = \frac{1}{l} \qquad 2.60$$

This can be solved exactly by letting $u = A + B\cos(\Delta\theta)$

Doing the math gives $\Delta = \sqrt{1-6\left(h/cl\right)^2}$, from which we can directly determine the orbital precession.

$$\left(\frac{h}{cl}\right)^2 = \frac{GM}{c^2 l}\ .$$

For the planet mercury $l = 55.443\times10^6 \quad km$

$$\frac{GM}{c^2} = 1.475 \quad km$$

Giving $\Delta = 0.999999920188298314$

The precession per revolution in radians is $2\pi - 2\pi\Delta = 5.014717113745\times10^{-7} \quad radians$

Converting to arc seconds the precession becomes: 0.103435965

Mercury orbits the sun 414.9378 times in one earth century, so the precession per century is: 42.9195 arc seconds per century in excellent agreement with observation and GR.

Conclusions

The precession observed in the orbit of Mercury can be explained with nothing more then well accepted physical formulas and relatively simple mathematics. No assumptions about the nature of the gravitational field are required and space is treated as "flat". A curious term appears in the calculations that deserve some exploration. The k_1 term is orbital path related. It skews the gravitational constant and this skewing can be positive or negative or zero depending on the eccentricity of the orbit. For the Pioneer space probe e=1.7372 and k_1 is positive resulting in an increase in the gravitational force. Prior to being subjected to the "sling shot effect" its eccentricity was most likely much smaller maybe even negative. This would have resulted in an effective increase in G after it reached its escape velocity.

I am not familiar with the orbital data of the craft but welcome any input from anyone who has more information.

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