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Investigation of the fundamental equations of quantum mechanics in terms of the special theory of relativity

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Abstract: The fundamental equations of quantum physics, which describe the motion of a free particle, are analyzed by taking into account the special theory of relativity. A relativistic relation between the energy and momentum of a quantum particle is defined and it is shown that the straightforward usage of the non-relativistic equations creates a contradiction in the group velocity. The correct relationships for the non-relativistic case are put forward by considering the relativistic relation between the energy and momentum.

Key Words: Classical mechanics, Dynamics, Kinematics, Relativity, Special relativity, Quantum mechanics

1. INTRODUCTION

The quantum mechanics has its roots on the paper of Planck, in which he solved the problem of radiation by a black body by using a heuristic approach [1]. Planck proposed the electromagnetic radiation could be the sum of the discontinuous energy packets, called quanta. This suggestion was the beginning of our understanding on the dualistic nature of light. Four years later Einstein used the concept of the quanta in order to explain the photoelectric effect [2]. The important equation of

$$E = \hbar w \tag{1}$$

was put forward by these papers. This equation also represents the relation between the corpuscular and wave behaviors of light. E and w expresses the energy and angular frequency, respectively. $\hbar$ is the angular constant of Planck. In some cases energy (E) is shown by the difference of two energy levels as $W_1 - W_2$ [3]. The second important development in the quantum theory was the works of de Broglie [4, 5]. He proposed the phenomena of the wave particle duality for quantum particles like electrons and neutrons. De Broglie also introduced the equation of

$$p = \hbar k \tag{2}$$

for p and k are the momentum and wave-number, respectively. After these inventions, which were the corner stones of the quantum theory, the general theory for exploring the behavior of quantum particles was developed by the works of Schrödinger and the Copenhagen school. Schrödinger introduced a differential equation by using the Hamiltonian analogy with Eqs. (1) and (2) [6, 7]. Since he considered the non-relativistic relations of mechanics, his equation was for the non-relativistic case. The relativistic version, named as the Klein-Gordon equation, was developed in the same years [8]. Another approach for the formalism of quantum mechanics was the matrix formulation, developed by the members of the Copenhagen school (especially by Heisenberg) [9]. Schrödinger showed that the matrix formalism was in analogy with his equation [10]. As a result we can conclude that the foundations of quantum theory lay at the consideration of the non-relativistic equations of motion and energy with Eqs.(1) and (2).

It is the aim of this paper to examine the non-relativistic equations of motion and energy approximating from the relativistic relations of energy and momentum. First of all we will derive a relation, which is also valid in the non-relativistic cases, between the relativistic energy and momentum of a particle. Then investigate the well-known non-relativistic equations of kinetic energy with momentum and check if they satisfy the relation, derived in the relativistic case. We will propose more general expressions of these non-relativistic equations and interrogate the present usage of them in quantum mechanics. This paper is based on the ideas, developed in Refs. [11].

2. THEORY

The special theory of relativity was put forward by Einstein in order to examine the behavior of particle systems at velocities that sufficiently approach to the velocity of light [12]. The equations of energy and momentum can be given by

$$E = mc^2 \tag{3}$$

and

$$p = mv_g \tag{4}$$

where *m* is the relativistic mass of the particle which can be written as

$$m = \frac{m_0}{\sqrt{1-\alpha}}. \tag{5}$$

c and v_g are the velocity of light and the group velocity, respectively. m_0 is the rest mass of the particle. α is equal to v_g / v_p for $v_g v_p$ gives c^2. The definitions of the group and phase velocities are utilized according to the works of de Broglie [4, 5]. Now we will introduce a relation between the energy and momentum as

$$\frac{E}{p} = v_p \tag{6}$$

by using Eqs. (3) and (4) with the property of $v_g v_p = c^2$. It is apparent that this equation is also valid and must be satisfied for the non-relativistic case.

The non-relativistic energy equation [13] for a free particle can be written as

$$E = \frac{1}{2} m_0 v_g^2. \tag{7}$$

The non-relativistic case is defined by the approximation of $v_g << c$. The momentum of the same particle is given by

$$p = m_0 v_g. \tag{8}$$

If we attempt to divide the relation of energy, given by Eq. (7), by the momentum, we will obtain the equation of

$$\frac{E}{p} = \frac{v_g}{2} \tag{9}$$

which does not satisfy Eq. (6). Equation (6) can also be represented by

$$\frac{E}{p} = \frac{c^2}{2v_p} \tag{10}$$

according to the relation of $v_g v_p = c^2$. Up to here, beginning from Eq.(7), we have dealt with the classical mechanics. We can examine the quantum physical aspects of the subject by using Eqs.(1) and (2). The relation of

$$\frac{w}{k} = v_p \tag{11}$$

can be evaluated from Eqs.(1), (2) and (6). Equation (11) can also be expressed as

$$\frac{w}{k} = \frac{c^2}{v_g} \tag{12}$$

according to the relation of $v_g v_p = c^2$. One obtains

$$v_g = c\sqrt{2} \tag{13}$$

by using Eqs. (9) and (12). It is obvious that Eq. (13) contradicts with the non-relativistic condition of $v_g << c$. At this point we can conclude that **1)** Eq. (8) can not be used with Eq. (7)

and **2)** the equations, defined on relying of the common usage of Eqs. (8) and (7) must be reconsidered.

The reason of the mistake, which occurs by the usage of Eq. (8) with Eq. (7), can be put forward by taking into account the relativistic equations, given by Eqs. (3) and (4). The non-relativistic relation of energy can be obtained by

$$E \approx (T_1 + T_2)c^2 \tag{14}$$

for $v_g << c$. T_1 and T_2 are equal to

$$T_1 = m_0 \tag{15}$$

and

$$T_2 = \frac{1}{2} m_0 \frac{v_g^2}{c^2} \tag{16}$$

respectively. The same approximation can be made for the momentum as

$$p = (T_1 + T_2)v_g \, . \tag{17}$$

The ratio of Eq.(14) to Eq. (17) gives directly Eq. (6). We can rewrite Eq. (9) as

$$\frac{E}{p} = \frac{T_2}{T_1} \frac{c^2}{v_g} \tag{18}$$

by taking into account these points. The error that yields the contradiction, mentioned by Eq. (13), can be understood directly from Eq. (18). The wrong terms are divided. It is obvious that if we consider Eq. (7) as the energy relationship for a quantum particle, then we must use the equation of $p = T_2 v_g$ for the momentum. Since Eq.(7) is taken into account for the derivation of the Schrödinger equation [13], we will redefine the relationship between the angular frequency and wave-number. The non-relativistic energy can be expressed by Eq. (7). We can write the relation of

$$\frac{1}{2} m_0 v_g^2 = \frac{2p^2}{m_0} \frac{c^4}{v_g^4}. \tag{19}$$

Energy can be defined by

$$E = \frac{2p^2}{m_0 \alpha^2} \tag{20}$$

by putting Eq. (19) in Eq. (7). The relation of dispersion can be obtained as

$$w = \frac{2\hbar k^2}{m_0 \alpha^2} \tag{21}$$

when Eqs. (1) and (2) are used in Eq. (20). Equation (21) can be rewritten as

$$w = \frac{\hbar k^2}{T_2 \alpha}. \tag{22}$$

It is apparent that Eq. (21) satisfies the condition, given by Eq. (6) and does not cause a contradiction. In a similar way, we can define the relationship of dispersion for relativistic cases by taking into account Eqs. (3) and (4). A relation can be introduced as

$$mc^2 = \frac{p^2}{m\alpha} \tag{23}$$

which leads to the equation of

$$E = \frac{p^2}{m\alpha} \tag{24}$$

when used in Eq. (3). As a result one obtains

$$w = \frac{\hbar k^2}{m\alpha} \tag{25}$$

for a relativistic mater wave.

3. CONCLUSION

In this paper we dealt with the non-relativistic equations of energy and momentum since they are the fundamental relations of quantum physics when considered with Eqs.(1) and (2). It is shown that the direct usage of Eq. (7) directly with Eq. (8) yields erroneous results and its reason is explained by considering the non-relativistic approximations of the relativistic equations of energy and momentum. Also the dispersion relations, which are important in the definition of wave packets and the derivation of the Schrödinger equation, are introduced. It is mentioned that these equations satisfy the condition, given by Eq. (6).

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