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Kriging Approach to Incompatibility

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Abstract

Kriging is parent theory of the least squares. This paper presents kriging approach to the problem of incompatibility in the field of the Einstein Theory of Relativity.

Let us consider (e.g. for $j = n+1$) variance of the difference of two random variables V_j and $\hat{V}_j$, where $E\{V_j\} = E\{\hat{V}_j\} = m = 0$, in the terms of covariance

$$\begin{aligned} D^2\{V_j - \hat{V}_j\} &= Cov\{(V_j - \hat{V}_j)(V_j - \hat{V}_j)\} \\ &= Cov\{V_j V_j\} - Cov\{V_j \hat{V}_j\} - Cov\{\hat{V}_j V_j\} + Cov\{\hat{V}_j \hat{V}_j\} \\ &= Cov\{V_j V_j\} - 2Cov\{\hat{V}_j V_j\} + Cov\{\hat{V}_j \hat{V}_j\} \end{aligned}$$

introducing the estimation statistics $\hat{V}_j = \sum_i \omega_j^i V_i = \omega_j^i V_i$

$$\begin{aligned} D^2\{V_j - \hat{V}_j\} &= Cov\{V_j V_j\} - 2Cov\{\hat{V}_j V_j\} + Cov\{\hat{V}_j \hat{V}_j\} \\ &= Var\{V_j\} - 2Cov\{\textstyle\sum_i \omega_j^i V_i V_j\} + Cov\{(\sum_i \omega_j^i V_i)(\sum_i \omega_j^i V_i)\} \\ &= \sigma^2 - 2\textstyle\sum_i \omega_j^i Cov\{V_i V_j\} + \sum_i \sum_l \omega_j^i \omega_j^l Cov\{V_i V_l\} \\ &= \sigma^2 - 2\sigma^2 \omega_j^i \rho_{ij} + \sigma^2 \omega_j^i \rho_{ii} \omega_j^i \ , \end{aligned}$$

where ρ_{ii} (to simplify notation) is (e.g. for $i = 1, \ldots, n$)

$$\underbrace{\begin{bmatrix} \rho_{11} & \ldots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \ldots & \rho_{nn} \end{bmatrix}}_{n \times n} = \underbrace{\begin{bmatrix} \rho(|1-1|) & \ldots & \rho(|1-n|) \\ \vdots & \ddots & \vdots \\ \rho(|n-1|) & \ldots & \rho(|n-n|) \end{bmatrix}}_{n \times n}$$

symmetric $n \times n$ matrix of stationary correlations.

The minimization constraint

$$\frac{\partial D^2\{V_j - \hat{V}_j\}}{\partial \omega_j^i} = 0$$

produces n equations in n unknowns ω_j^i

$$\underbrace{\begin{bmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & \rho_{nn} \end{bmatrix}}_{n\times n} \cdot \underbrace{\begin{bmatrix} \omega_j^1 \\ \vdots \\ \omega_j^n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} \rho_{1j} \\ \vdots \\ \rho_{nj} \end{bmatrix}}_{n\times 1}$$

equivalent to time-domain Wiener-Hopf equation in the set of weights ω_j^i

$$\sigma^2 \rho_{ii} \omega_j^i = \sigma^2 \omega_j^i \rho_{ii} = \sigma^2 \rho_{ij}$$

or frequency domain Wiener-Hopf equation in the transfer function $H(f)$

$$S_{ii}(f) H(f) = H(f) S_{ii}(f) = S_{ij}(f)$$

multiplied by ω_j^i

$$\omega_j^i \rho_{ii} \omega_j^i = \rho_{ij} \omega_j^i$$

and substituted into

$$D^2\{V_j - \hat{V}_j\} = E\{[V_j - \hat{V}_j]^2\} - (\underbrace{E\{V_j\}}_{0} - \underbrace{E\{\hat{V}_j\}}_{0})^2$$

gives the so-called 'mean squared error' minimized at every single j-th value

$$E\{[(V_j - m) - (\hat{V}_j - m)]^2\} = \sigma^2(1 - \rho_{ij}\omega_j^i) = \sigma^2(1 - \rho_{ji}\omega_j^i) = \sigma^2 \rho_{jj} - \sigma^2 \rho_{ji} \omega_j^i \ ,$$

where

$$\rho_{ij} = \rho(|i-j|) = \rho(|-(i-j)|) = \rho(|j-i|) = \rho_{ji} \ ,$$

or at unit value of f

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} P(f) df = \int_{-\frac{1}{2}}^{+\frac{1}{2}} [S_{jj}(f) - S_{ij}(f)^* H(f)] \, df = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left[S_{jj}(f) - \frac{|S_{ij}(f)|^2}{S_{ii}(f)} \right] df$$

cause the variance of the estimation statistics is minimized at every single j-th value

$$E\{[\hat{V}_j - m]^2\} = Cov\{\hat{V}_j \hat{V}_j\} = \sigma^2 \omega_j^i \rho_{ii} \omega_j^i = \sigma^2 \rho_{ji} \omega_j^i$$

or at unit value of f

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} Q(f) df = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left[\frac{|S_{ij}(f)|^2}{S_{ii}(f)} \right] df \ .$$

Since outcoming of input value (e.g. for $j = n$)

$$\hat{V}_j = \sum_{i=1}^{n} \omega_j^i V_i = \sum_{i=1}^{n} \delta_j^i V_i$$

is unknown

$$E\{[\hat{V}_j - m]^2\} = \sigma^2 \rho_{ij} \delta_j^i = \sigma^2$$

its mean squared error can not be equal to zero value then the so-called 'mean squared error'

$$E\{[(V_j - m) - (\hat{V}_j - m)]^2\} = \sigma^2(1 - \rho_{ij}\delta_j^i) = 0$$

is not a mean squared error.

Since only for infinite set of outcome values v_i of V_i

$$\lim_{n\to\infty} \sum_{i=1}^{n} \omega_j^i v_i = \lim_{n\to\infty} \sum_{i=1}^{n} \omega_j^i V_i = m = 0$$

then each (finite) weighted average seen as the outcome value of the statistics

$$\hat{V}_j = \sum_{i=1}^{n} \omega_j^i V_i \quad \text{or} \quad \hat{V}_u = \sum_{i=1}^{n} \omega_j^i V_{u-i}$$

is the conditional mean with zero (for the Wiener filter) mean value as central value of its error distribution and variance of the estimation statistics as its mean squared error

$$E\{[\hat{V}_j - m]^2\} = \sigma^2 \rho_{ij} \omega_j^i$$

or

$$E\{[\hat{V}_u - m]^2\} = \sigma^2 \rho_{ij} \omega_j^i = \text{const.}\ .$$

If linear trend (non-stationary and non-zero mean) is taking into consideration

$$\hat{V}_j = \hat{\epsilon}_j + \beta^k f_{kj} = \hat{\epsilon}_j + \beta^1 \underbrace{f_{1j}}_{1} + \beta^2 \underbrace{f_{2j}}_{j}$$

and

$$V_i = \epsilon_i + \beta^k f_{ki} = \epsilon_i + \beta^1 \underbrace{f_{1i}}_{1} + \beta^2 \underbrace{f_{2i}}_{i} \quad i = 1, \ldots, n$$

the unbiasedness condition

$$E\{\hat{V}_j\} = E\{\omega_j^i V_i\}$$

produces two equations

$$f_{kj} = f_{ki}\omega_j^i$$

the minimization constraint produces n equations in $n+2$ unknowns $\omega_j^1, \ldots, \omega_j^n, \mu_j^1, \mu_j^2$

$$\rho_{ii}\omega_j^i + f_{ki}\mu_j^k = \rho_{ij}$$

multiplied by ω_j^i

$$\omega_j^i \rho_{ii} \omega_j^i = -f_{kj}\mu_j^k + \omega_j^i \rho_{ij}$$

gives the minimized mean squared error

$$E\{[\hat{V}_j - \beta^k f_{kj}]^2\} = \sigma^2 \omega_j^i \rho_{ii} \omega_j^i = \sigma^2(\rho_{ij}\omega_j^i - f_{kj}\mu_j^k)$$

associated to kriging weights

$$\omega_j^i = \rho^{ii}\rho_{ij} - \rho^{ii} f_{ki}\mu_j^k \ ,$$

where

$$\mu_j^k = A^{kk} f_{ki}\rho^{ii}\rho_{ij} - A^{kk} f_{kj} \quad \text{and} \quad A^{kk} = (f_{ki}\rho^{ii} f_{ki})^{-1} \ .$$

The non-linear BLUE equation (the mean squared error identical to zero value)

$$\rho_{ij}\omega_j^i - f_{kj}\mu_j^k = 0 \quad \text{if} \quad \rho_{ij} \neq \vec{0}$$

has only asymptotic least-squares approximation for $j \to \infty$

$$\rho_{ij}\omega_j^i - f_{kj}\mu_j^k \approx \rho_{ij}\omega_j^i + \frac{1}{2} f_{kj} A^{kk} f_{kj} \approx 2\xi \ ,$$

where

$$\rho_{ij}\omega_j^i \approx \xi \approx \frac{1}{2} f_{kj} A^{kk} f_{kj} \ ,$$

then

$$\rho_{ij}\omega_j^i - \frac{1}{2} f_{kj} A^{kk} f_{kj} \approx 0$$

associated to asymptotic least-squares weights

$$\omega_j^i \approx \rho^{ii} f_{ki} A^{kk} f_{kj} \ .$$

The mean squared error if the correlations vanish

$$\sigma^2 f_{kj} A^{kk} f_{kj} = \frac{\sigma^2 (j^2 - 2m_n j + m_{sn})}{n\sigma_n^2} \quad \text{if} \quad \rho_{ij} = \vec{0}$$

associated to the least-squared weights

$$\omega_j^i = \rho^{ii} f_{ki} A^{kk} f_{kj}$$

can be identical to zero value (incompatibility)

$$f_{kj} A^{kk} f_{kj} = 0 \Leftrightarrow j = m_n \pm I\sigma_n \ ,$$

where: $m_n = \bar{i}$, $m_{sn} = \overline{i^2}$, $\sigma_n = \sqrt{\overline{i^2} - \bar{i}^2}$, $I = \sqrt{-1}$.

Let us consider Robertson-Walker metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + g_{ll}(dx^l)^2 \ ,$$

where: $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$,

$$g_{11} = \frac{a(t)^2}{1 - Kr^2} \ , \qquad g_{22} = a(t)^2 r^2 \ , \qquad g_{33} = a(t)^2 r^2 \sin^2\theta \ .$$

Solving the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu}$$

we get

$$G_{ll} = -\left(\frac{K + a'(t)^2 + 2a(t)a''(t)}{a^2(t)}\right) g_{ll} \quad \text{and} \quad G_{00} = 3\left(\frac{K + a'(t)^2}{a(t)^2}\right) \ .$$

Now, we can rewrite the metric

$$ds^2 = -dt^2 - \left(\frac{G_{00}}{3} + \frac{2a''(t)}{a(t)}\right)^{-1} G_{ll}(dx^l)^2$$

for $G_{00} = 8\pi \mathsf{G}\varrho$ and $G_{ll} = 8\pi \mathsf{G} p g_{ll}$ equal to

$$ds^2 = -dt^2 - 8\pi \mathsf{G} p \left(\frac{8\pi \mathsf{G}\varrho}{3} + \frac{2a''(t)}{a(t)}\right)^{-1} g_{ll}(dx^l)^2$$

identical to

$$ds^2 = -dt^2 + g_{ll}(dx^l)^2$$

if

$$\frac{a''(t)}{a(t)} = -\frac{4\pi \mathsf{G}}{3}(\varrho + 3p)$$

and if

$$p \neq 0 \Leftrightarrow R_{ll} - \frac{1}{2}g_{ll}R \neq 0 \ .$$

Following kriging approach to incompatibility, the Einstein equation, like

$$\rho_{ij}\omega^i_j - \frac{1}{2}f_{kj}A^{kk}f_{kj} \approx 0 \ ,$$

is only approximation for $p \approx 0$

$$R_{ll} - \frac{1}{2}g_{ll}R = g_{ll}Z - \frac{1}{2}g_{ll}R \approx 0$$

of the parent equation that if the spatial components of the metric vanish for $p = 0$, like the mean squared error if the correlations vanish

$$f_{kj}A^{kk}f_{kj} = 0 \ ,$$

can be identical to zero value (incompatibility)

$$R(t) = \frac{6[K + a'(t)^2 + a(t)a''(t)]}{a^2(t)} = 0 \quad \text{if} \quad p = 0 \quad \text{and} \quad g_{00}R(t) = -R(t) = 0 \quad \text{if} \quad \varrho = 0$$

associated to

$$Z(t) = \frac{2K + 2a'(t)^2 + a(t)a''(t)}{a^2(t)} \quad \text{if } p = 0 \quad \text{and} \quad Z(t) = \frac{3a''(t)}{a(t)} \quad \text{if } \varrho = 0$$

then

$$a(t)Z(t) + a''(t) = 2\left[\frac{K + a'(t)^2}{a(t)} + a''(t)\right] = 0 \quad \text{and} \quad \frac{3}{a(t)}\left[\frac{K + a'(t)^2}{a(t)} + a''(t)\right] = 0$$

solving the differential equation inside the braces

$$a(t) = \frac{C}{\sqrt{K + a'(t)^2}} \ ,$$

where C is a constant, substituting $\tan\alpha = a'(t)$ for $K = 1$ the parametric solution is $a = C\cos\alpha$ and $t = C(1 - \sin\alpha)$, substituting $\coth\beta = a'(t)$ for $K = -1$ the parametric solution is $a = C\sinh\beta$ and $t = C(\cosh\beta - 1)$, for $K = 0$ we get $a(t) = \sqrt{2Ct}$.

References

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