

Flux quantisation of a superconductor within a gauge field theory of scale relativity

Sameer Al-Khawaja

AECS, Damascus, Syria

Email: skhawaja@aec.org.sy

Copyright © Sameer Al-Khawaja

Abstract

The aim of this article is to extend the principles of gauge field theory founded by Nottale in the framework of the scale relativity theory, to the quantum mechanical electric current density associated with a multiply connected superconductor. Nottale's gauge transformation which is identified with a scale transformation of the internal resolution, and assumed to be a global dilation, is employed to recast the expression of magnetic flux threading the superconductor. Due to the inherent fractal nature of the geometry postulated by the scale relativity theory, an additional term representing the "state of scale"; the ratio of the relative internal resolutions, modulates the well-known quantised flux relation. We emphasise that this modulation is issued as a natural ramification of the *a priori* assumption that charges are built from the symmetries of the scale space, which the scale relativity predicts.

1 Introduction

Different frameworks for compromising the theory of relativity and quantum mechanics at the fundamental level have been propounded. Amongst these is the theory of scale relativity, which has been established and advanced by Nottale and co-workers[1]. In this theory, Einstein's principle of relativity in motion laws is generalised to scale transformations, so that one may reconsider the problem of fundamental scales in physics. Thus the postulated notion of the axiomed differentiability of space-time coordinates is renounced in such an approach. One can instead identify space-time resolutions characterising the state of the reference system, by resolution transformations. In addition, the equations governing the system are required to preserve their form under such resolution transformations i.e., they must be scale covariant. Despite the validity of the concept of differentiability in classical mechanics, this hypothesis does not stand when considering quantum mechanical paths of micro-objects, as firstly illustrated by Feynman[4]. In the context of scale relativity theory, the geometric objects are considered of being continuous but non-differentiable of particular topological dimension. A measure of these

objects such as length, area or volume is, therefore, explicitly dependent on the resolution ϵ at which they are considered. This measure tends to infinity when the resolution interval goes to zero, implying a non-differentiable fractal space-time continuum. Having a constraint on this scale-dependence presumption, one may then make physical quantities dependent on space-time resolutions. That is, covariance of equations under transformations of resolutions is sustained, in a similar manner to motion covariance[5]. We shall extend the gauge invariance proposed on scale relativity to the quantum mechanical current density to derive the flux expression of a superconductor in multiply connected geometry. It is worth noting here that the validity of scale relativity has been discussed elsewhere, and the reader is referred to published works on the subject as in [2, 6, 7]. Before proceeding a brief prelude to scale relativistic theory of electromagnetism is presented.

2. Gauge field theory on the basis of scale relativity

2.1 Electric charges and electromagnetic field

The scale relativity theory imposes a scaling limit above which the internal fractal structures of the trajectories of a particle are smeared out. The limit represents a transition scale related to the Compton length $\lambda = h/mc$ of the particle under consideration [6, 8], i.e., to its inertial mass. In the quantum domain, these structures are most relevant and identified in a relative way. As an instance, in the electromagnetic theory only the relative scale $\eta = \lambda/\epsilon$ has a physical meaning that is equivalent to the relativity of scales tacit in the relativity equations of motion. In other words, the displacement of the particle in space-time is linked to a change of the scale of a given structure of the fractal trajectories. Since the scale-space is presumably considered non-absolute, the scale of this structure will not be the same at different positions of the ensemble of trajectories representing the particles. In scale relativity, this relative scale-space is designated by the internal space of standard gauge theory, as shown by Nottale [7-8]. One may write the scale ratio as being explicitly dependent on the space-time coordinates, i.e., $\eta = \eta(x, y, z, t)$, hence in a displacement of the particle, the relativity of scales implies that the resolution at which this structure appears in the new location will in principle be different from the initial one. Therefore, the occurrence of resolution change due to the fractal geometry in the form of dilations is probable and is induced by translations that read [2, 8]

$$q \frac{\delta \mathcal{E}}{\mathcal{E}} = -A_{\mu} dx^{\mu} \quad (1)$$

where q is the electric charge and the field A_{μ} can be identified with the electromagnetic potential φ . Invoking the ratio $\eta = \lambda/\epsilon$, one may write Eq. 1 in the form

$$\delta \ln \eta = \frac{1}{q} A_{\mu} dx^{\mu} \quad (2)$$

Introducing the Coulomb electric potential $\varphi = q/r$, we may observe that the division of A_{μ} on the

charge q is justified as $\ln \eta$ is dimensionless and therefore only a charge-independent and purely geometric contribution has a role in Eq. 2. Since the scale-space is considered to be non-absolute, it is anticipated that the scale of a structure will change throughout a displacement of the particle in space-time. Thus one may recover the inertial part of the variation in resolution by subtracting the term related to the geometric effects of curvature from the total variation according to a scale-covariant derivative.

Thus setting $\xi = q \ln \eta$, one may write the dilation field as [8]

$$D\xi = d\xi - \delta\xi = d\xi - A_\mu dx^\mu \quad (3)$$

It is worthwhile to note here that the effect of fractality starts with scalars. Finally, we obtain according to partial covariant derivative the sum of the inertial and geometric terms:

$$\partial_\mu \xi = D_\mu \xi + A_\mu \quad (4)$$

By definition the action S of the electron is related to the Lagrangian L and may be expressed on the basis of a relativity principle from the space-time invariant s such that

$$dS = -mcds \quad (5)$$

so that the least action principle $\delta \int dS = 0$ allows geodesics motion of the electron. The geodesical curves shall then be a function of the scale variable ξ at scales below λ , due to the fractal nature of the curves, i.e., $S = S(\xi)$. One may thus write the following differential making use of Eq. 3:

$$dS = \frac{\partial S}{\partial \xi} d\xi = \frac{\partial S}{\partial \xi} (D\xi + A_\mu dx^\mu) \quad (6)$$

Consequently we obtain

$$\partial_\mu S = D_\mu S + \frac{\partial S}{\partial \xi} A_\mu \quad (7)$$

Nottale demonstrated that the term $\partial S / \partial \xi = -e/c$ represents the passive charge on which the electromagnetic field acts (see references [1] and [2]). Charges are therefore, built from the symmetries of the scale space, which is defined in the internal relative resolution. Thus one may write

$$dS = -mcds - \frac{e}{c} A_\mu dx^\mu \quad (8)$$

Equation 8 implies a novel geometric interpretation of the particle-electromagnetic field interaction that is, the increase of the length is caused firstly by the contribution of the usual variation owing to motion of the particle. Secondly, it comes from the new geometric contribution which is essentially a length dilation of the internal fractal structure. The problem becomes a matter of coupling of a charged particle to the electromagnetic field; this is described as an energy-momentum transfer between the motions. Since the latter interaction involves the "external" geometry (in the general relativistic sense) and the internal geometry, it worthwhile to construct the four-dimensional energy-momentum tensor. The 4-velocity vector is given by [9]

$$u^\alpha = \left(\frac{1}{\sqrt{1-v^2/c^2}}; \frac{v}{c\sqrt{1-v^2/c^2}} \right) \quad \alpha = 0,1,2,3 \quad (9)$$

which is tangent to the geodesic of a particle. The 4-force vector is also expressed as

$$g^\alpha = \left(\frac{\mathbf{F}v}{c^2\sqrt{1-v^2/c^2}}; \frac{\mathbf{F}}{c\sqrt{1-v^2/c^2}} \right) \quad (10)$$

where \mathbf{F} is a force. Hence the fundamental equations of relativistic dynamics can be rewritten in the form

$$g^\alpha = \frac{dp^\alpha}{ds} \equiv mc \frac{du^\alpha}{ds} \quad (11)$$

For the interacting electron with the electromagnetic field in scale relativity we can write a geodesic equation minimising the length-invariant [10] (the proper time) by virtue of the least-action principle

$\delta \int dS = 0$. Expressing the field by the anti-symmetric tensor $F_{\alpha\mu} = \partial_\alpha A_\mu - \partial_\mu A_\alpha$, the least-action principle results in the Lorentz equation of electrodynamics

$$mc \frac{du^\alpha}{ds} = \frac{e}{c} F_{\alpha\mu} u^\mu \quad (12)$$

Equation 8 can be formulated to yield the differential of the action as a function of the coordinates such as

$$dS = -(mcu_\mu + \frac{e}{c}A_\mu) dx^\mu \quad (13)$$

Electrodynamics of a quantum particle and gauge invariance

The action S as implied by Eq. 12, is complex since it is dependent on the complex 4-velocity \mathbf{V}^μ accounting for the scalar particle, i.e., $S = S(x^\mu, \mathbf{V}^\mu, \eta)$. Hence Eq. 12 may be written in the form as follows

$$dS = -mc\mathbf{V}_\mu dx^\mu - \frac{e}{c}A_\mu dx^\mu \quad (14)$$

On the other hand, the wave function is expressed in terms of S according to the relation

$$\psi = |\psi| e^{i\frac{S}{\hbar}} \quad (15)$$

Consequently one can write

$$dS = -i\hbar d \ln \psi = -mc\mathbf{V}_\mu dx^\mu - \frac{e}{c}A_\mu dx^\mu \quad (16)$$

Therefore, the relation between the wave function and the velocity reads

$$mc\mathbf{V}_\mu = i\hbar D_\mu \ln \psi = i\hbar \partial_\mu \ln \psi - \frac{e}{c} A_\mu \quad (17)$$

The resultant well-known QED-covariant derivative

$$-i\hbar D_\mu = -i\hbar \partial_\mu + \frac{e}{c} A_\mu \quad (18)$$

coincides with the scale-covariant derivative dictated by Eq. 3, but it is acting in this case on the wave function, and Eq. 17 takes the form

$$D_\mu = \partial_\mu + i \frac{e}{\hbar c} A_\mu \quad (19)$$

If we now define a relative scale $\eta' = \lambda/\varepsilon'$ within which the electron lies [8], we can write a similar form to Eq. 2

$$\delta \ln \eta' = \frac{1}{q} A'_\mu dx^\mu \quad (20)$$

Identifying the ratio of both scales ε and ε' as χ , the Galilean scale relativity allows one to write

$$\chi = \frac{\eta'}{\eta} \quad (21)$$

so that we obtain

$$A'_\mu = A_\mu + q \partial_\mu \ln \chi \quad (22)$$

The latter equation identifies a gauge transformation with a scale translation of the internal resolutions in the form of a global dilation. Thus the wave function of the particle transforms and becomes

$$\psi' = \psi \exp(-i \frac{eq}{\hbar c} \ln \chi) \quad (23)$$

where the amplitude of ψ has been arbitrarily set to unity. The gauge transformation leaves the Lagrangian given by Eq. 14 invariant. For an electron $q = e$, the relation $e^2 = 4\pi\alpha\hbar c$ relates both the electronic charge and α , the fine structure constant, which defines the coupling between the electron and the electromagnetic field. Thus Eq. 23 becomes

$$\psi' = \psi \exp(-i 4\pi\alpha \ln \chi) \quad (24)$$

For a superconducting material $q = 2e$, Eqs. 22 and 24 can be then rewritten as

$$A'_\mu = A_\mu + 2e \partial_\mu \ln \chi \quad (25)$$

$$\psi' = \psi \exp(-i 8\pi\alpha \ln \chi) \quad (26)$$

3. Application of scale relativistic gauge transformation to superconductivity

The BCS theory of superconductivity is established fundamentally on the interaction between a

superconductor and the electromagnetic field. This interaction in the frame of QED, occurs via the energy exchange of electrons with opposite spins and momenta to form the electron pairs of charge $q = 2e$. The resulting pairs therefore, acts like bosons since each pair has a net spin of zero [11], and these bosons by means of Bose-Einstein condensation process undergo a transition to form the macroscopic superconducting state. The latter state is a quantum mechanical ground state, described by a macroscopic single-particle wave function $\psi = |\psi| e^{i\theta}$. It is important to recognise that the phase θ in the wave function plays a significant role in the formation of the superconducting state, which is associated with a spontaneous symmetry breaking. The phase symmetry of the wave function is broken when ascribing a particular phase to define the ground state. Conventional QED gauge transformations have been used to arrive at magnetic flux quantisation in a ring-shaped superconductor [11]. A superconducting ring dubbed a thick ring (of thickness much greater than the penetration depth), is topologically multiply connected and by virtue of the macroscopic quantum mechanical nature of the condensate state with charge $q = 2e$ it was originally demonstrated that in an applied magnetic field, magnetic flux can be trapped within the ring as it is cooled through the superconducting transition temperature. The flux quantisation expression $\Phi = n \Phi_0$ where n is integer and $\Phi_0 = \pi hc / e$ is the flux quantum, is normally derived by choosing a closed path inside the bulk superconducting ring whereby the included flux is given by

$$\Phi = \int A_\mu dx^\mu \quad (27)$$

Since the wave function ψ is invariant when its phase changes by an integer multiple of 2π , from the quantum mechanical probability current density the aforementioned flux quantisation is then ensued. In the following derivation of the formula of magnetic flux threading the fractal superconducting ring, we make use of a gauge transformation that takes into account the scale translation of the internal resolutions as implied by the relations (22) and (23). The probability current density is quantum mechanically given by [12]

$$j_\mu(\psi) = \frac{-iq\hbar}{2m^*} (\psi^* (\partial_\mu \psi) - \psi (\partial_\mu \psi)^*) \quad (28)$$

For a superconductor the charge and mass are $q = 2e$, $m^* = 2m_e$ respectively.

Under an electromagnetic field the operator ∂_μ ought to be modified to include the relativity of scales, and will then have the form

$$\partial_\mu = \partial_\mu - \frac{iq}{\hbar c} \ln \chi A_\mu \quad (29)$$

Substituting (28) into the current density (27) we get

$$j_\mu(\psi) = \frac{-iq\hbar}{2m^*} [\psi^* (\partial_\mu - \frac{iq}{\hbar c} \ln \chi A_\mu) \psi - \psi (\partial_\mu + \frac{iq}{\hbar c} \ln \chi A_\mu) \psi^*] \quad (30)$$

Acting on the wave function $\psi = |\psi| e^{i\theta}$, where $\theta = -(ieq/hc) \ln \chi$, one obtains from (30)

$$j_{\mu}(\psi) = \frac{-iq\hbar}{2m^*} |\psi|^2 [2\partial_{\mu}(i\theta) - 2\frac{iq}{\hbar c} \ln \chi A_{\mu}] \quad (31)$$

Substituting for q and m^* the expression (31) finally yields

$$j_{\mu}(\psi) = \hbar \partial_{\mu}\theta - \frac{2e}{c} \ln \chi A_{\mu} \quad (32)$$

Since the screening current is essentially zero and only significant close to the surface of the ring, then deep inside the bulk ring around a closed contour Γ we require

$$\int_{\Gamma} j_{\mu}(\psi) dx^{\mu} = 0 \quad (33)$$

Thus from (32) one obtains

$$\hbar \int_{\Gamma} \partial_{\mu}\theta dx^{\mu} = \frac{2e}{c} \int_{\Gamma} \ln \chi A_{\mu} dx^{\mu} \quad (34)$$

By virtue of (27) we can write

$$\frac{2e}{c} \int_{\Gamma} \ln \chi A_{\mu} dx^{\mu} = \frac{2e}{c} \ln \chi \Phi \quad (35)$$

where Φ is the magnetic flux entering the ring. In addition, the left-hand side of (34) may be given by

$$\hbar \int_{\Gamma} \partial_{\mu}\theta dx^{\mu} = \hbar(\Delta\theta) \quad (36)$$

The invariance of the wave function implies that the change in θ is equal to integer multiple of 2π i.e., $\Delta\theta = n(2\pi)$. Thus the equality (34), with the aid of (35) and (36), yields

$$\hbar (2\pi n) = \frac{2e}{c} \ln \chi \Phi$$

which gives

$$\Phi = \frac{n}{\ln \chi} \Phi_0 \quad ; \quad n = 0, \pm 1, \pm 2, \dots \quad (37)$$

where $\Phi_0 = \pi\hbar c / e$ is the flux quantum as mentioned above.

4. Discussion and conclusion

What the result stemming from the magnetic flux expression (36) entails is that a supplemental term, which represents the relative "state of scale" χ appears in the quantification of flux. This scale ratio develops primarily in the phase of the wave function of the associated charged particles (Cooper pairs), after a gauge transformation to the "field" A_{μ} has been applied. In addition, since within the frame of scale relativity, we adopt scales that lie well below the De Broglie scale λ , which represents the

bifurcation point in the transition to the fractality of the quantum space-time of microphysics[7], then

is encapsulated and becomes dominant in the flux Φ . When $\chi = \eta' / \eta = 2.17828128$ we can recover the relation $\Phi = n \Phi_0$, which is obtainable by the standard gauge transformations. The latter value of χ is slightly above the fractal dimension $D_f = 2$ of the trajectories of quantum particles, and at this value of χ one may not retain fractality of flux.

References

- [1] Nottale L, 1995, Scale Relativity, Fractal Space-Time and Quantum Mechanics in: Quantum Mechanics, Diffusion and Chaotic Fractals, El Naschie M. S., Rössler O. E. and Prigogine I. (eds.), (Elsevier, Oxford,), 51-77.
- [2] Nottale L., 1996, Chaos, Solitons and Fractals **7**, 6, 877.
- [3] Nottale L., 1999, Chaos, Solitons and Fractals **10**, 2-3, 459.
- [4] Feynman R. P. and Hibbs A. R., 1965, Quantum Mechanics and Path Integrals, New York, McGraw-Hill.
- [5] Weinberg S., 1972, Gravitation and Cosmology, New York, Wiley and Sons.
- [6] Nottale L., 2001, Chaos, Solitons and Fractals **12**, 1577.
- [7] Nottale L., 1993, Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity, Singapore, World Scientific.
- [8] Nottale L., Célérier M. N. and Lehner T., arXiv: hep-th/ 0307093 v1 10 July 2003.
- [9] Eddington A. S., 1990, The Mathematical Theory of Relativity, Cambridge University Press.
- [10] Lifshitz E. M., 1995, The Classical Theory of Fields, New York, Pergamon.
- [11] Tilley D. R. and Tilley J., 1990, Superfluidity and Superconductivity, 3rd Edition, London, IoP Publishing Ltd.
- [12] Schiff Leonard I., 1968, Quantum Mechanics, 3rd Edition, Singapore, McGraw-Hill.