

Generalized Lorentz Coordinate Transformation

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ABSTRACT

This paper derives equations for generalized Lorentz coordinate transformation based on a simple principle that only the wave equation must be form invariant and valid in both coordinate systems. The usual requirement that the speed of light is a universal constant and thus has an identical value in any inertial coordinate systems is removed. This leads to some interesting conclusions and possible new interpretations of the Special Relativity Theory.

INTRODUCTION

The Special Relativity Theory (SRT) is one of the fundamental theories of modern physics with applications in many branches of science starting from the Solid State Physics, Chemistry, Optics, to Quantum Physics etc. The everyday use of the GPS navigation system, which is also firmly based on SRT and in particular on the Lorentz Coordinates Transformation (LCT)^[1,2], confirms its validity. With increasing accuracy of modern instrumentation the theory can be considered verified many times over to a high degree of accuracy. Thus it seems that the SRT is complete and that nothing new can be added to it even though the interpretation is still somewhat mysterious.

In this paper a possible generalization of the LCT is presented with similar conclusions as in the Edward's theory^[3] that can be used to improve our understanding and interpretation of SRT. The LCT can be derived from a simple assumption that the speed of light is constant and independent of inertial motion of either the source of the light or the observer. The independence on the speed of the light source is easily understandable by assuming that the light is propagating in a medium, aether, where the

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velocity is constant, but the independence on the speed of the observer is puzzling. The independence on the speed of the source is also clearly supported by the everyday observations of stars and planets. It would not be possible to see and measure the orbits of the stars or planets and confirm the Kepler's laws if the speed of light would depend on the speed of the source. The arrival times of various observations would be vastly different particularly for very distant objects and an agreement with the Kepler's laws would not be possible.

In this paper it will be shown that the two coordinate systems in relative inertial motion can be connected by a coordinate transformation but in each coordinate system the speed of light or for that matter the speed of any signal propagation, can be different. It is well known that the speed of light in a gravitational field is reduced from its standard vacuum value, so the derived GLCT can be used to connect the vacuum coordinate system with the moving coordinate system in a gravitational field.

DERIVATION OF THE COORDINATE TRANSFORMATION

As is typical in similar derivations of the LCT, the two coordinate systems, the primed, which is moving and the unprimed, laboratory coordinate system, which is stationary, are considered. This is for clarity illustrated in Fig.1.

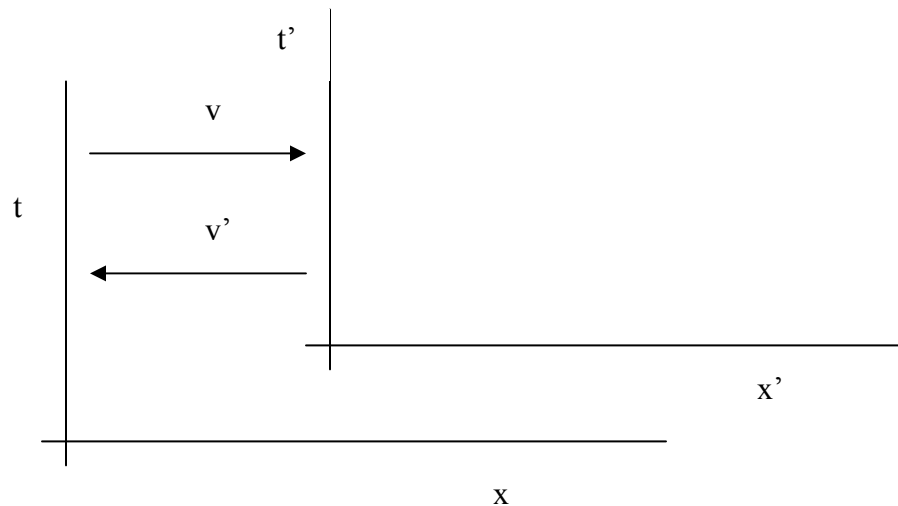


Fig.1. Schematic diagram of the laboratory coordinate system and the moving, primed, coordinate system. The velocity “ v ” observed in the laboratory coordinate system is different from the velocity v' observed from the moving coordinate system.

In this derivation it will be assumed that the coordinate transformation shown below:

$$x = x(x', t', c', v/c), \quad (1)$$

$$t = t(x', t', c', v/c), \quad (2)$$

is such that in the primed coordinate system a wave equation for an arbitrary physical quantity φ is satisfied and has the following form:

$$\frac{1}{c'^2} \frac{\partial^2 \varphi}{\partial t'^2} - \frac{\partial^2 \varphi}{\partial x'^2} = 0. \quad (3)$$

However, it will be assumed that the propagation velocity c' is not necessarily identical with the propagation velocity c in the laboratory coordinate system. If the inertial motion of the primed coordinate system affects all the physical parameters of the measurement including the distances and time, we have no way to internally assert that the speed of light measured in one system is identical with the value measured in the other. The universal constancy of the speed of light when going from one coordinate system to the other is only an assumption. In the derivation below it will be assumed that the speed of the propagation as appears in Eq.3 is not necessarily maintained. In order to show this it is necessary to derive all the partial derivatives needed in Eq.3 and substitute them there. For the first partial derivatives it holds:

$$\frac{\partial \varphi}{\partial t'} = \frac{\partial \varphi}{\partial x} \frac{\partial x}{\partial t'} + \frac{\partial \varphi}{\partial t} \frac{\partial t}{\partial t'}, \quad (4)$$

$$\frac{\partial \varphi}{\partial x'} = \frac{\partial \varphi}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial \varphi}{\partial t} \frac{\partial t}{\partial x'}. \quad (5)$$

The formulas for the second derivatives are more complex, but also relatively easy to evaluate as follows:

$$\frac{\partial^2 \varphi}{\partial t'^2} = \frac{\partial^2 \varphi}{\partial x^2} \left(\frac{\partial x}{\partial t'} \right)^2 + 2 \frac{\partial^2 \varphi}{\partial t \partial x} \frac{\partial t}{\partial t'} \frac{\partial x}{\partial t'} + \frac{\partial \varphi}{\partial x} \frac{\partial^2 x}{\partial t'^2} + \frac{\partial^2 \varphi}{\partial t^2} \left(\frac{\partial t}{\partial t'} \right)^2 + \frac{\partial \varphi}{\partial t} \frac{\partial^2 t}{\partial t'^2}, \quad (6)$$

$$\frac{\partial^2 \varphi}{\partial x'^2} = \frac{\partial^2 \varphi}{\partial x^2} \left(\frac{\partial x}{\partial x'} \right)^2 + 2 \frac{\partial^2 \varphi}{\partial t \partial x} \frac{\partial t}{\partial x'} \frac{\partial x}{\partial x'} + \frac{\partial \varphi}{\partial x} \frac{\partial^2 x}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial t^2} \left(\frac{\partial t}{\partial x'} \right)^2 + \frac{\partial \varphi}{\partial t} \frac{\partial^2 t}{\partial x'^2}. \quad (7)$$

After substitution into Eq.3 the following result is obtained:

$$\begin{aligned} & \frac{1}{c'^2} \frac{\partial^2 \varphi}{\partial t'^2} \left\{ \left(\frac{\partial t}{\partial t'} \right)^2 - c'^2 \left(\frac{\partial t}{\partial x'} \right)^2 \right\} - \frac{\partial^2 \varphi}{\partial x^2} \left\{ \left(\frac{\partial x}{\partial x'} \right)^2 - \frac{1}{c'^2} \left(\frac{\partial x}{\partial t'} \right)^2 \right\} + \\ & + \frac{\partial^2 \varphi}{\partial t \partial x} \left(\frac{2}{c'^2} \frac{\partial t}{\partial t'} \frac{\partial x}{\partial t'} - 2 \frac{\partial t}{\partial x'} \frac{\partial x}{\partial x'} \right) + \frac{\partial \varphi}{\partial t} \left\{ \frac{1}{c'^2} \frac{\partial^2 t}{\partial t'^2} - \frac{\partial^2 t}{\partial x'^2} \right\} + \frac{\partial \varphi}{\partial c} \left\{ \frac{1}{c'^2} \frac{\partial^2 x}{\partial t'^2} - \frac{\partial^2 x}{\partial x'^2} \right\} = 0 \end{aligned} \quad (8)$$

To obtain a wave equation in the laboratory coordinate system where the propagation speed is c the following relations must be satisfied:

$$\left(\frac{\partial t}{\partial t'}\right)^2 - c'^2 \left(\frac{\partial t}{\partial x'}\right)^2 = \frac{c'^2}{c^2}, \quad (9)$$

$$\left(\frac{\partial x}{\partial x'}\right)^2 - \frac{1}{c'^2} \left(\frac{\partial x}{\partial t'}\right)^2 = 1, \quad (10)$$

$$\frac{1}{c'^2} \frac{\partial t}{\partial t'} \frac{\partial x}{\partial t'} = \frac{\partial t}{\partial x'} \frac{\partial x}{\partial t'}, \quad (11)$$

$$\frac{1}{c'^2} \frac{\partial^2 t}{\partial t'^2} - \frac{\partial^2 t}{\partial x'^2} = 0, \quad (12)$$

$$\frac{1}{c'^2} \frac{\partial^2 x}{\partial t'^2} - \frac{\partial^2 x}{\partial x'^2} = 0. \quad (13)$$

Substituting Eq.9 and Eq.10 into Eq.11 squared the following result is obtained:

$$\frac{1}{c'} \frac{\partial x}{\partial t'} = \varepsilon_1 c \frac{\partial t}{\partial x'}, \quad (14)$$

where an index function ε was introduced to keep track of the signs after taking the square root of the result. Similarly, from Eq.9, Eq.10 and Eq.14 it is possible to obtain the result:

$$\frac{c}{c'} \frac{\partial t}{\partial t'} = \varepsilon_2 \frac{\partial x}{\partial x'}. \quad (15)$$

By differentiating Eq.14 and 15 it is easily seen that Eq.12 and Eq.13 are satisfied when the index functions satisfy the following condition:

$$\varepsilon_1 \varepsilon_2 = 1. \quad (16)$$

For the sake of simplicity it will be assumed that both index functions are equal to $\varepsilon_i = 1$.

As a result we can write the following three equations that need to be satisfied:

$$\frac{1}{c'} \frac{\partial x}{\partial t'} = c \frac{\partial t}{\partial x'}, \quad (17)$$

$$\frac{c}{c'} \frac{\partial t}{\partial t'} = \frac{\partial x}{\partial x'}, \quad (18)$$

$$\left(\frac{\partial t}{\partial t'}\right)^2 - c'^2 \left(\frac{\partial t}{\partial x'}\right)^2 = \frac{c'^2}{c^2} \quad (19)$$

From Eq.17 and Eq.18 it directly follows that the transformation equations, Eq.1 and Eq.2, must be linear functions of their coordinate arguments. It is therefore reasonable to seek the solution for the transform in the following form:

$$x = \gamma(x' + \alpha t'), \quad (20)$$

$$ct = \gamma(c't' + \beta x'). \quad (21)$$

The result is as follows:

$$x = \frac{1}{\sqrt{1 - \beta^2}} (x' + \beta c't'), \quad (22)$$

$$ct = \frac{1}{\sqrt{1 - \beta^2}} (c't' + \beta x'), \quad (23)$$

with only one remaining parameter β that needs to be determined from the condition that the velocity with which the primed coordinate system moves is measured in the laboratory coordinate system and is equal to “v”. The final result for the Generalized LCT then becomes:

$$x = \frac{1}{\sqrt{1 - v^2/c^2}} \left(x' + \frac{v}{c} c't' \right), \quad (24)$$

$$ct = \frac{1}{\sqrt{1 - v^2/c^2}} \left(c't' + \frac{v}{c} x' \right), \quad (25)$$

with the corresponding inverse as follows:

$$x' = \frac{1}{\sqrt{1 - v'^2/c'^2}} \left(x - \frac{v'}{c'} ct \right), \quad (26)$$

$$c't' = \frac{1}{\sqrt{1 - v'^2/c'^2}} \left(ct - \frac{v'}{c'} x \right). \quad (27)$$

It is also obvious that the following relation must hold:

$$\frac{v}{c} = -\frac{v'}{c'} \quad (28)$$

In Eq.24 and Eq.25 it is interesting; however, to replace the laboratory coordinate system variables on the right had side of equations by the primed coordinate system variables. It can thus be considered that the laboratory coordinates should be found from the velocity

and the speed of light values known only internally to the primed coordinate system. This becomes:

$$x = \frac{1}{\sqrt{1 - v'^2 / c'^2}} (x' - v' t'), \quad (29)$$

$$t = \frac{c'}{c} \frac{1}{\sqrt{1 - v'^2 / c'^2}} \left(t' - \frac{v'}{c'^2} x' \right). \quad (30)$$

The resulting formulas thus become identical to the standard LCT formulas with the only exception being the time scaling factor c'/c in Eq.30. It is also necessary to note that v' is negative in the primed coordinate system. From this result it becomes apparent that the time in the laboratory coordinate system cannot be determined without knowing the speed of light in that system.

For completeness the relativistic formula for compounding of velocities becomes as follows:

$$\frac{u}{c} = \frac{\frac{u'}{c'} - \frac{v'}{c'}}{1 - \frac{u'v'}{c'^2}}. \quad (31)$$

where u' is again the velocity measured internally in the primed coordinate system.

When it is considered that the light velocities are identical in both coordinate systems the formulas revert to their traditional form. It is important to note, however, that all the expressions contain only the ratios of velocities to the speed of light in the given coordinate system. So, it is possible to conclude that only the relative velocities referenced to the speed of light is what matters. In reality there may not be any reliable measurement available for obtaining an absolute velocity internally in a particular inertial coordinate system, so the assumption about the universal constancy of the speed of light may not be verifiable. The measurement of the velocity in each coordinate system in actuality may be only a comparison of the velocity to the speed of light in that coordinate system. When any particular measurement technique is analyzed in detail it may be found that every clock in the inertial coordinate system with no gravitational forces is actually based on some signal propagation phenomenon. So, we may only think that we are measuring time, but instead we are comparing velocities.

Since the light speed in the primed coordinate system is not determined from the above derivations it is possible to select it such that it corresponds to the common perception. For example, it is possible to consider that for the one way speed of light ^[4] we have the relation:

$$c' = c - v. \quad (32)$$

This choice leads to the following transformation:

$$x = \frac{1}{\sqrt{1 - v^2/c^2}} \left(x' + v \left(1 - \frac{v}{c} \right) t' \right), \quad (33)$$

$$t = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\left(1 - \frac{v}{c} \right) t' + \frac{v}{c^2} x' \right). \quad (34)$$

It is now only necessary to synchronize the clock in the moving system such that:

$$\left(1 - \frac{v}{c} \right) t' = t'', \quad (35)$$

and the standard LCT equations are obtained. For the two way measurement the first order corrections cancel and we obtain the standard LCT equations without any change to the clock synchronization.

The GLCT can now be also used to explain very simply the Sagnac Effect ^[5]. The speed of light in the direction of the trajectory rotation and the direction of the counter trajectory rotation are different. The resulting pulse arrival time difference in the rotating coordinate system is thus as follows:

$$\Delta t' = l' \left(\frac{1}{c - v} - \frac{1}{c + v} \right) = \frac{4\pi R v}{c^2 - v^2}, \quad (36)$$

where R is the radius of the circular rotating light trajectory. It can also be shown that due to the centrifugal force resulting from the rotation, the clock on the periphery of the rotating trajectory and the clock in the laboratory run with the same rates. It thus holds that $\Delta t' = \Delta t$. The same is true for the so-called Ehrenfest paradox ^[6]. The length of the circumference is not changed by the rotation, the SRT length contraction is compensated to the second order in “v” by the centrifugal force effect, so it holds true that $l' = 2\pi R$.

Finally, it is interesting to derive the transformation invariant, the metric line element ds , as follows:

$$ds^2 = (cdt)^2 - dx^2. \quad (37)$$

After differentiating Eq.24 and Eq.25 and substituting into Eq.37 the result becomes:

$$ds^2 = (c' dt')^2 - dx'^2 = ds'^2. \quad (38)$$

The coordinate transformation invariant is maintained in both coordinate systems. Therefore, in principle, all the equations of SRT are valid using the GLCT. The only significant difference is the fact that if the motion of the observer's coordinate system is in some sort of aether and causes the change in the observer's system speed of light, it cannot be internally detected. All other physical parameters that depend on the speed of light will automatically adjust and internally no change will be noticed.

CONCLUSIONS

In this article it was shown that the Lorentz Coordinate Transformation can be generalized in such a way that it is not necessary to consider the universal constancy of speed of light. The SRT and all its conclusions are not affected by this generalization. The only significant change is in interpretation. It is now possible to consider the aether again and change in the moving observer's speed of light as his system moves through it. The observer in the moving coordinate system, however, cannot internally measure this difference and cannot notice any change since all his time measurements are referenced to his local speed of light.

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