Relativistic Pendulum and the Weak Equivalence Principle

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ABSTRACT

This paper derives equations for the relativistic proper period of oscillations of a pendulum driven by the electrical forces and for a pendulum driven by the gravitational forces. The derivations are based on the Einstein's Special Relativity Theory and in particular on the Lorentz coordinate transformation, which has been experimentally verified many times and which is a well-recognized principle for all the modern physics. Since the pendulum proper period of oscillations is an absolute inertial motion invariant the derived formulas may be used to study the motion dependence of the inertial and gravitational masses. It is found that the well-publicized equivalence between these two masses, which is assumed independent of any inertial motion, cannot be sustained and a new mass equivalence principle must be considered where the equivalence of these two masses holds only at rest.

INTRODUCTION

The pendulum is an ages proven device that has attracted attention of many researchers in the past for its simplicity of operation, its accuracy to measure time, and for its ability to study the gravitational or electrical fields. One can only wonder why it was not studied in modern times in more detail, since it offers some clues for resolving the "mystery" of the inertial and gravitational mass equivalence, the so called Einstein's week equivalence principle [1]. Recently an interesting article was published [2] where the author derived relativistic equations of motion for the pendulum starting from a simple relativistic Lagrangian and the formula for the relativistic conservation of energy. This paper will also focus its attention of the relativistic equations of motion of the simple pendula, one that is driven by electrical forces, and the second one that is driven by gravitational forces and will compare how these pendula behave when they undergo an inertial motion relative to the laboratory coordinate system. The key idea of this work is to derive formulas for the proper period of oscillations of the particular pendulum in terms of the pendulum physical parameters such as the mass of the bob, the length of the pendulum string, and the remaining parameters of the experimental setup. Since the proper period of oscillations is an inertial motion invariant, the derived formulas must thus also be inertial motion invariants and

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this will force some interesting and perhaps unexpected inter-relations between the parameters of the experimental setup.

ELECTRICAL PENDULUM

The experimental setup for the pendula experiments is shown in Fig 1. For the electrical pendulum case it is assumed that the bob is nearly an ideal mass point, has a negligible diameter, has an inertial rest mass m_o, and is charged by charge $-q_o$. The string has the rest length l_o, it is not conductive, has a negligible mass and a very large stiffness, so it does not change its length during the swing of the bob. The string is anchored at the point "A" without any friction or any other mechanical resistance. Finally, the pendulum motion will be considered sufficiently slow that no radiation effects from the charge or mass acceleration will have to be considered.

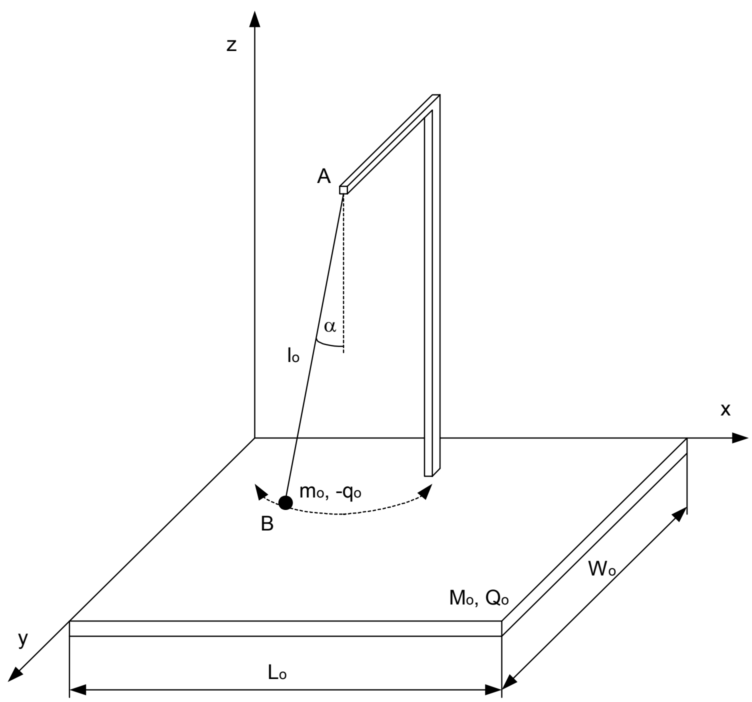

Fig.1. Pendulum positioned in a laboratory coordinate system x,y,z that is anchored above a very large base plate to make the vertical electrical or gravitational fields above the plate uniform and where the fringing field effects from the edges can be neglected. The base plate has the area $A_o = L_oW_o$ and may have a gravitational mass M_o, or charge Q_o. The gravitational filed of the Earth is not considered here. The bob has inertial mass m_o and may have charge $-q_o$.

The base plate which supports the pendulum via a very stiff, mass-less and non conductive beam arrangement is sufficiently large, so that the electrical field above the plate can be considered vertical and uniform within the pendulum swing path. The area of the base plate is $A_o = L_oW_o$. The base plate is also nonconductive and charged by a total charge Q_o that is embedded into the plate's matrix and cannot move. The angle of the pendulum swing is determined by the angle α that will be considered very small in these experiments. With these assumptions it is then possible to write for the electrical filed the following:

$$E = \frac{Q_o}{\varepsilon 2 A_o} . \tag{1}$$

This result follows directly from the Maxwell's electrical field divergence equation and the Gauss integral formula. If the bob were stationary then the vertical force on the bob would simply be the field multiplied by charge q. However, the bob is moving and therefore producing a current, which carries with it a magnetic filed. This filed then interacts with the charge of the base plate and generates additional force according to the Lorentz force equation. The computation of this force may not be so simple for an arbitrary velocity of the bob and for an arbitrary angle, so it is desirable to make some plausible simplifications. Since the charge is a universal invariant it is reasonable to expect that the force on the charge will be the same if we consider the bob steady and the base plate moving in opposite direction or when we consider the base plate steady and the bob moving, at least for the case when the angle α is zero or near zero. It is thus easy to see that the moving plate produces a magnetic filed along the y direction with the intensity as follows:

$$B_y = \frac{\mu \cdot Q_o v}{2A_o\sqrt{1 - v^2/c^2}} . \tag{2}$$

The vertical component of the force acting on the bob will then be according to Lorentz equation:

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{3}$$

equal to:

$$F_z = \frac{-q_o \cdot Q_o}{\varepsilon 2A_o\sqrt{1 - v^2/c^2}}\left(1 - \frac{v^2}{c^2}\right). \tag{4}$$

In these equations it was considered that the length L_o undergoes the Lorentz contraction, v is the bob instant velocity, $\varepsilon \cdot \mu = c^{-2}$, and c is the speed of light. This force can now be made equal to force referenced to the laboratory coordinate system, which is not moving even though the result was obtained when it was momentarily considered that the coordinate system is referenced to the bob, which is moving, since charge is a motion invariant as was already mention above. In the next step the pendulum motion equation will be derived using Newton's second law written in the relativistic form. This can be expressed as follows:

$$\frac{d}{dt}\left(\frac{m_o \vec{v}}{\sqrt{1-v^2/c^2}}\right)=\vec{F}\ . \tag{5}$$

By considering that the tangential velocity of the bob is: $v = l_o d\alpha/dt$ it is possible using Eq.4 to write for a small α the following result:

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\frac{d\alpha}{dt}\right)=\frac{-q_o Q_o\sqrt{1-v^2/c^2}}{\varepsilon\cdot l_o m_o 2A_o}\cdot\alpha\ . \tag{6}$$

By introducing the proper time $d\tau = dt\sqrt{1-v^2/c^2}$, Eq.6 is simplified as follows:

$$\frac{d^2\alpha}{d\tau^2}+\frac{q_o Q_o}{2\varepsilon\cdot l_o m_o A_o}\cdot\alpha = 0\ . \tag{7}$$

From this result then directly follows that the proper period of the pendulum oscillations for small deflections α is equal to:

$$\tau_o = 2\pi\sqrt{\frac{2\varepsilon\cdot l_o m_o A_o}{q_o Q_o}}\ . \tag{8}$$

Since the left hand side of Eq.8 is an invariant, the right hand side must also be an invariant and we can thus construct the pendulum invariant $I_o = l_o m_o A_o$. This quantity must be independent of any inertial motion. For example: when the whole pendulum contraption, including the base plate, moves uniformly in the x direction with the speed u_x the result for the pendulum invariant will be:

$$I(u_x) = l_o\frac{m_o}{\sqrt{1-u_x^2/c^2}}A_o\sqrt{1-u_x^2/c^2} = I_o\ . \tag{9}$$

Indeed the pendulum invariant will remain unchanged even when the laboratory observer will measure a different bob inertial mass and a different base plate length L. The pendulum string will be observed having the same length as in the rest. The same result is obtained for the motion in the z direction where the plate area is not changed but the string will be observed with a contracted length. This result thus applies to any uniform motion without rotation in any direction. This result is not surprising, it is expected, and it also follows directly from the classical Newtonian physics by replacing the standard time with the proper time in the equation for the oscillation period. This result can be further generalized to any clock powered by electrical fields and having bobs with an inertial mass. For example a mechanical wristwatch, or quartz watch will all work the same way, since they use the mass inertia and the electrical fields (converted to a mechanical spring action) somewhere in their system.

Finally this result can be considered as yet another way to show that the inertial mass must depend on velocity as follows:

$$m_i = \frac{m_o}{\sqrt{1 - v^2 / c^2}} \quad . \tag{10}$$

GRAVITATIONAL PENDULUM

The experimental setup for the gravitational pendulum is the same as for the electrical pendulum. The only difference is that the bob and the base plate are not charged; instead the base plate has now a large gravitational mass M_o. The gravity of the Earth will not be considered here. It will be assumed that the plate supplies all the gravitational force for the pendulum. To be consistent in the derivations, similar equations for the gravitational filed as the Maxwell's field equations will be used for this case. M.L. Ruggiero and A. Tartaglia[3] have published linearized Einstein's field equations for the small velocities and obtain the following system:

$$\vec{\nabla} \cdot \vec{E}_g = -4\pi \cdot \kappa \cdot \rho_g \,, \tag{11}$$

$$\vec{\nabla} \times \frac{1}{2} \vec{B}_g = -\frac{1}{c} \cdot 4\pi \cdot \kappa \cdot \vec{j}_g \,, \tag{12}$$

where the E_g and B_g are the gravitostatic and the gravitomagnetic fields respectively and the ρ and j are the mass density and the mass current density. From these equations it is again simple to find the solutions for the configuration that is used in this experiment and arrive at the force equation relative to the bob as follows:

$$F_z = \frac{-2\pi \cdot \kappa \cdot m_o \cdot M_o}{A_o (1 - v^2 / c^2)} \left(1 - 2\frac{v^2}{c^2} \right) \,. \tag{13}$$

In the derivation it was considered that the gravitational mass M_o obeys the same velocity dependence as the inertial mass according to the weak equivalence principle. The inertial mass is not an invariant, as in the previous case of the electrical pendulum, where the charge was, and consequently, when the transform of this force is made back to the laboratory coordinate system, it is necessary to divide this result by another Lorentzian factor $\sqrt{1 - v^2 / c^2}$. The equation of motion for the gravitational pendulum will thus finally become:

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1 - v^2 / c^2}} \frac{d\alpha}{dt} \right) = \frac{-2\pi \cdot \kappa \cdot M_o}{l_o A_o \left(1 - v^2 / c^2\right)^{\frac{3}{2}}} \left(1 - 2\frac{v^2}{c^2} \right) \cdot \alpha \,. \tag{14}$$

By introducing again the proper time into this equation the result will be:

$$\frac{d^2\alpha}{d\tau^2} + \frac{2\pi\cdot\kappa\cdot M_o}{l_o A_o}\frac{\left(1-2\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)^2}\cdot\alpha = 0 . \quad (15)$$

It is worth noting that the result does not depend on the mass of the bob. This is important for the confirmation of the fact that the result agrees with the Galileo's free fall experiments. Since the linearization of the Einstein's field equations was performed only for relatively small velocities, it is without problem to consider that the brackets containing the velocity factors actually cancel. This is reasonable, since the proper period of pendulum oscillations should not depend on the pendulum velocity. This could also be considered as a consistency check for the derivation as well as for the linearization of the Einstein's field equations. The equation for the proper pendulum oscillation period can thus be simply written as:

$$\tau_o = 2\pi\sqrt{\frac{l_o A_o}{2\pi\cdot\kappa\cdot M_o}} . \quad (16)$$

This is again an expected result that can be obtained directly from the Newtonian physics by replacing the standard time with the proper time. The gravitational pendulum invariant then becomes:

$$I_o = \frac{l_o A_o}{M_o} . \quad (17)$$

Unfortunately, when a test is made to see if this is really an invariant, a serious problem is encountered. For example, for an inertial motion in the x direction this will be:

$$I(u_x) = \frac{l_o A_o\sqrt{1-u_x^2/c^2}}{\frac{M_o}{\sqrt{1-u_x^2/c^2}}} = I_o\left(1-\frac{u_x^2}{c^2}\right). \quad (18)$$

The gravitational pendulum invariant is not an invariant!? The only reasonable conclusion that can be drawn from this result, as it was drawn previously for the inertial mass, is that the gravitational mass must depend on velocity as follows:

$$M_g = M_o\sqrt{1-v^2/c^2} . \quad (19)$$

This leads to a new mass equivalence principle $m_i m_g = m_o^2$. The Einstein's weak mass equivalence principle holds only at rest. To complete the proof it is necessary, however, to make certain that the derivation above is self-consistent. It is necessary to repeat the derivation again using the new mass equivalence principle, since the old weak mass equivalence principle was used in the derivation of Eq.13. It should also be noted that the above-derived dependency of the gravitational mass on velocity does not alter the fact that the massive bodies follow the geodetic curves in a curved space-time as is assumed in General Relativity Theory.

PENDULUM USING THE NEW MASS EQUIVALENCE PRINCIPLE

The experimental setup for this case is, of course, identical with the previous one. However, no gravitostatic and gravitomagnetic fields will be considered. The only assumption that will be used is that the gravitating mass depends on velocity according to Eq.19. With this assumption it is not necessary to consider the coordinate system in reference to the bob. It is sufficient to stay in the laboratory coordinate system and write for the equation of motion simply:

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\frac{d\alpha}{dt}\right)=\frac{-2\pi\cdot\kappa\cdot M_o\sqrt{1-v^2/c^2}}{l_o A_o}\cdot\alpha\,. \tag{20}$$

The square root term on the right hand side of Eq.20 remains there after the bob's gravitational mass dependence on velocity was substituted there according to Eq.19. By introducing the proper time again the result becomes:

$$\frac{d^2\alpha}{d\tau^2}+\frac{2\pi\cdot\kappa\cdot M_o}{l_o A_o}\cdot\alpha=0\,. \tag{21}$$

No simplifications and no approximations, except the small angle oscillations assumption are necessary to obtain this result. The gravitation pendulum invariant is the same as before:

$$I_o=\frac{l_o A_o}{M_o}, \tag{22}$$

except that it is now the true invariant independent of any inertial motion similarly as in the electrical pendulum case. It is comforting to know that any clock being electrical or grandfather's type will keep the same time in any moving inertial coordinate system. It would certainly be exceedingly strange if the electrical clock and the gravity driven clock displayed a different time when in motion.

Finally, from Eq.10 it also directly follows that:

$$m_i^2c^2-m_i^2v^2=m_o^2c^2\,. \tag{23}$$

This formula can be converted to a well know four-vector energy-momentum equation:

$$\frac{E^2}{c^2}-p^2=m_o^2c^2, \tag{24}$$

using the famous Einstein's mass-energy equation:

$$E = m_i c^2 \; . \qquad (25)$$

However, from the above derivation it is clear that this equation is strictly valid only for the inertial masses and not for the gravitational masses, since it is derived only for that case. The extension to gravitational masses may not be fully justified, or the validity of it is just a coincidence since Eq.24 is not satisfied by the gravitational masses. Some other physical phenomenon may be responsible for the validity of Einstein's mass-energy equation in nuclear mass to energy conversions. An indication that such a phenomenon may be related to gravity can be found in the relativistic Lagrangian. It is well know that Lagrangian for a free massive particle has the form:

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \; . \qquad (26)$$

By identifying the right hand side of Eq.26 with the gravitational mass-energy the Lagrangian is simply:

$$L = -m_g c^2 \; . \qquad (27)$$

Since the Lagrangian is a difference of the kinetic and potential energy in the classical limit it is clear that at rest the Lagrangian will contain only a potential energy of the system thus corresponding to the gravitational rest mass generated by the process, hence also the minus sign. In motion, however, after applying the Lorentz coordinate transformation, the same Lagrangian will also contain a kinetic energy. The equation of motion is, of course, easily obtained as an Euler-Lagrange equation corresponding to the action integral:

$$\delta \int_{t_1}^{t_2} - m_g c^2 dt = 0 \, . \qquad (28)$$

This will lead to:

$$\frac{d}{dt}\left(\frac{-\partial c^2 m_g}{\partial v} \right) = \frac{d}{dt}\left(\frac{m_o v}{\sqrt{1 - v^2 / c^2}} \right) = 0 \;\; , \qquad (29)$$

which finally implies that:

$$\frac{v}{c} m_i = -\frac{\partial m_g}{\partial \left(\frac{v}{c} \right)} \, . \qquad (30)$$

This equation provides a clear link of the gravitational mass to the inertial mass and confirms again that the inertial mass cannot be equal to the gravitational mass when in motion.

DISCUSSION

The immediate objection to this result, even though the logic of the derivation is indisputable, is that the weak mass equivalence principle has already been experimentally proven. But let's see if this is really so. It seems that all the experiments and the theoretical arguments prove only Eq.10 and Eq.24 as for example in the Compton effect or in particle accelerators. There is only one experimental evidence, which may be considered a good counter argument against the validity of Eq.19, and these are the atomic weights of the elements. This is the only case where the masses are really weighed in a gravitational filed of the Earth. However, it seems that to assume the validity of Eq.10 automatically for the gravitational mass on the basis of the atomic weights is jumping to a conclusion. The processes in the nucleus are complex and other forces are in play. What if the nuclear forces present in the nucleus provide the gravitational mass to the nucleus when the energy and the inertial mass are supplied in a collision. For example, the well-known search for the Higgs boson might be just the right answer for this problem. The nucleus is not in an inertial motion, so this does not contradict Eq.19.

The arguments in support of Eq.19 are, on the other hand, many: According to Eg.19 photons do not have gravitational mass, so they cannot be confined by a black hole. More over, if they had the gravitational mass, they would certainly attract each other during the long trip from the distant galaxies and collapse into a single clump. There would be no galaxies to observe. The same applies to gravitons, if they exist, and neutrinos, which are assumed to have a small rest mass.

The famous formula for the deflection of light by a gravitating mass is calculated assuming that the relativistic line element of the space-time metric ds = 0. This also permits the assumption of zero gravitational mass. The Newton's classical calculation of the light deflection by a gravitating mass gives one half of the observed value. This is very strange and not logical, the result should be zero, since the Newton's theory considers only a flat space-time. It is well known that the light bending by gravity is a curved space-time phenomenon with a correct result obtained only when a curved space-time metric is considered. This discrepancy disappears when it is considered that light has no gravitational mass also in Newton's classical theory.

The pulsating stars such as the Cepheid variables can be considered as a natural gravity driven clocks, although not a purely gravity driven. The astronomers use them as a measuring yardstick for distance measurements in the universe. It is assumed that they keep a correct time even when they move away at some, not negligible, receding velocities. If Eq. 22 were not an inertial invariant large errors in the estimation of the universe distances would be obtained.

The expansion of the universe, for the distant galaxies that recede with high velocities, is less and less affected by the gravity, so the galaxies will not slow down. There is no need to consider any strange dark matter or a cosmological constant that generates a repulsive force.

The initial extremely fast expansion of the universe was not slowed down by gravity, since all the matter moved at a very high velocity as we can observe today on the distant galaxies that still carry it. The Universe would not be able to explode, since the extreme gravity would pull it immediately back to a small subatomic region.

The quantum vacuum generates many virtual particles that should have a gravitational mass and collapse the universe. This is not happening, because the fast moving virtual particles have no gravitational mass.

CONCLUSIONS

In this article it was shown that the weak equivalence principle is not valid for moving bodies. This was clearly illustrated by deriving the relativistic expression for the proper period of oscillations of a gravity driven pendulum. Based on these results a new mass equivalence principle was therefore proposed where the gravitational mass depends on the velocity differently than the inertial mass. Several arguments against and for the validity of the new equivalence principle were also briefly discussed.

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