

Time Evolution of the Precisely Localized Positive or Negative Energy State of the Free Dirac Electron

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Abstract

Precisely localized positive or negative energy states of the free Dirac particle are constructed without any arbitrary assumptions and shown to evolve with time in a complicated way under the action of the Foldy–Wouthuysen transformation of the free Dirac equation. “When the initial uncertainty in position is small on the scale of the Compton wavelength, there is an associated uncertainty in the mean energy that is large compared with the rest mass of the electron. However, this does not lead to any breakdown of the one-particle description, associated with the possibility of pair-production, but rather leads to” [1] time evolution of the precisely localized positive or negative energy states in the path-integral form, at the speed of the conventional velocity operator of the non-relativistic Pauli theory of free electron, without any acausal or superluminal spreading.

Introduction:

The two major physics discoveries of the first part of the last twentieth century, quantum mechanics and Einstein's theory of special relativity present new problems when treated together. Here, the first and foremost problem of localization in the relativistic quantum mechanics of a particle with nonzero rest-mass – “whether arbitrarily precise localization is possible, and if so, how it should be described – is almost as old as relativistic quantum mechanics itself [4, 5, 6, 7, 8, 9]. Despite the best efforts of many researchers over the intervening years [10, 11, 12, 13, 14, 15], the problem continues to attract much discussion [16], indicating that there is no general acceptance of any of the resolutions proposed to date.

A view sometimes expressed is that all the difficulties associated with the localization problem arise because any attempt to localize a particle on a scale small compared to its Compton wavelength, involves an uncertainty in energy so large that pair-production becomes possible, and a one-particle description of the physics becomes inconsistent.” [3] That is why, in 1929, Oskar Klein [17] stumbled into an apparent

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paradox while trying to describe the scattering of a relativistic free electron by a square potential using Dirac's relativistic wave equation. The Klein paradox results from our insistence in giving a single-particle interpretation to the relativistic Dirac wave function. Actually, a multi-particle interpretation of the relativistic Dirac equation explains the Klein paradox [18] by contemplating that the energy "uncertainty" introduced in quantum theory combines with the mass-energy equivalence of special relativity to allow the creation of particle/anti-particle pairs by quantum fluctuations when the theories are merged.

“As pointed out by Newton and Wigner [9] in their well-known paper on the problem, this view of multi-particle interpretation of the relativistic Dirac equation “really denies the possibility of the measurement of the position” of a particle. Some authors have considered it appropriate to abandon any attempt at one-particle localization, and to focus instead on local observables associated with quantized relativistic fields [15], but this approach has evidently failed to satisfy the many physicists who have continued to investigate the problem at the one-particle level [11, 12, 13, 15, 16]. Because the theory of relativity is, at its heart, a theory of relations between events in space-time, and because it is difficult to imagine what can constitute an event other than the instantaneous localization of a particle, the denial of one-particle localizability is hard to accept. It is important to see that the argument regarding pair-production and the Compton wavelength is inconclusive, because a single particle can have an arbitrarily large energy, and hence an arbitrarily large uncertainty in its energy. Therefore, the fact that a sufficiently small uncertainty in position implies an uncertainty in energy much greater than rest-mass energy does not *in itself* imply a breakdown of the one-particle picture.”[3] Moreover, starting with this one-particle picture in 1928, Dirac showed that if the wave function of a single free electron is set in such a way as to satisfy the requirements Einstein's theory of special relativity and quantum mechanics, the existence of the spin angular momentum and spin magnetic moment of that free electron can be deduced without any arbitrary assumptions. Likewise, in what follows we show that the precise localization of the single free electron also automatically follows in following steps by starting from the aforesaid Dirac wave function of the single free electron without any arbitrary assumptions.

- a. An infinite dimensional Hilbert space is defined for a free dirac particle such that the complete set of basis vectors satisfy the orthonormality property and the state of the free dirac particle, called as wave-function, is defined by a vector in this Hilbert space.
- b. This wave-function vector is then shown to be the general plane wave solution of the free dirac equation that is defined in the laboratory inertial reference frame at any arbitrary time instant.
- c. This general plane wave solution, obtained by using superimposition principle of quantum mechanics, of the free dirac equation is differentiated locally with respect to time to have the axial components of the linear momentum operator substituted with their eigenvalues in the resulting time-differentiated free dirac equation that is also defined in the laboratory inertial reference frame at the aforesaid arbitrary time instant.

- d. The resulting time-differentiated free Dirac equation is first multiplied on both sides by an arbitrary but fixed momentum eigen value exponential term and then, integrated globally over the entire volume space at the aforesaid arbitrary time instant in the laboratory inertial reference frame.
- e. The result of the integration in the preceding step is a free Dirac equation satisfied by the time-dependent amplitude of the plane wave corresponding to the aforesaid arbitrary but fixed momentum eigen value. Taking this arbitrariness of the momentum eigen value as generalized momentum co-ordinate defined in the momentum space, what we get out of the aforesaid integration is the second quantization of the single particle free Dirac Equation.
- f. This second quantized single particle free Dirac Equation is analytically solved in the momentum space for the Foldy-Wouthuysen (FW) representation. The analytical solution thus obtained is nothing but the description of the precise localization of the single free Dirac particle in the laboratory inertial reference frame at any arbitrary time instant.

The second problem of combining relativity and quantum mechanics is the Einstein causality that requires a finite propagation speed for all physical particles. Indeed, any solution of the free Dirac equation, whether it has positive energy or not, cannot propagate faster than with velocity of light. In non-relativistic Schrödinger equation, observables are represented by self-adjoint operators that, in the Heisenberg picture, depend on time. Therefore, measurements are localized in time but are global in space. The situation is radically different in the relativistic case. Because no signal can propagate faster than the speed of light, measurements have to be localized both in time and space. Causality demands then that two measurements carried out in causally-disconnected regions of space-time cannot interfere with each other. It is only when one insists in keeping a single particle interpretation of the relativistic Dirac equation that one crashes against causality violations. For instance, “in the Foldy-Wouthuysen (FW) representation of the single free Dirac particle, one expects a possible superluminal spreading of the wave-packets in the corresponding Hilbert space. Thus, if one takes the Newton Wigner operator (which is just the multiplication operator in the FW representation) as a position variable, then one have to live with an acausal propagation of initially localized single Dirac particle. This problem of initial position localization and subsequent acausality in the FW representation comes with all position operators commuting with the sign of energy.” [23]

In this paper, we show that the description of the precise localization of the single free Dirac particle in the laboratory inertial reference frame at any arbitrary time instant so arrived at in the step (f) above evolves in time in the form of a path-integral. Accordingly, we show in this paper that the twin problem of initial precise position localization and subsequent acausality in the FW representation is tackled in such a manner that the resulting time evolution of the path-integral of the initially precise position localized single Dirac free particle is without superluminal spreading in the laboratory inertial reference frame at any arbitrary time instant. In-fact, there are numerous proposals of the path integrals for a free Dirac particle in literature. A brief synopsis of these path integral methods is as follows:

“The path integral for the Dirac particle in one space dimension was found by Feynman [20, 21] and independently by Riazanov [26]. Even this relatively simple problem the path integral for the free Dirac particle in one space dimension, where there is no spin to complicate matters, is still attracting attention [27, 28]. In-fact, there is no consensus at all as to the form of path integrals for a free Dirac particle in three space dimensions. Previously proposed path integrals for a free Dirac particle in three space dimensions, fall into three broad categories. The first category [28, 29–36] comprises those approaches that work by “reverse engineering” introducing from the outset the Dirac matrices to describe the spin degrees of freedom. These approaches are often combined with the proper-time representation of the Dirac operator. Into the second category [37–39] fall those formulations that derive spin from continuously parameterized space of states related to the rotation group. To the third category [40–43] belong all approaches based on anti-commuting Grassmannian variables.”[2]

But the first two above categories of path integral methods lead to the same problem of superluminal propagation of the initially position localized free single Dirac particle in Foldy-Wouthuysen representation and the third category, based on anti-commuting Grassmannian variables, starts with the second quantization of the relativistic free Dirac equation & thereby completely rejects position localization of the single Dirac free particle at the outset itself. In this paper, we show without any arbitrary assumptions that the description of the precise localization of the single free Dirac particle in the laboratory inertial reference frame at any arbitrary time instant so arrived at in the step (f) evolves in time in the form of a new Dirac Kernel or propagator which is having contribution from infinite number of paths in momentum space. This result is unlike the corresponding result of the plane wave solutions of the free Dirac equation in literature wherein only one path contributes to the corresponding Kernel in momentum space. Also, we have demonstrated that for the free Dirac particle, the path integral between two fixed spatial points in the co-ordinate space during a certain time interval is composed, in time sequence, of a product of infinite number of the said new Dirac Kernels, each of which remains valid only for infinitesimal time duration at the intermittent spatial points on any path joining the aforesaid two fixed spatial points in the coordinate space and the individual retaining of new Dirac Kernels at the intermittent spatial points for infinitesimal time duration entails one to define a new position operator whose time derivative is the conventional velocity operator of the non-relativistic Pauli theory of free electron. Consequent upon the preceding demonstration, in the free time-evolution of the new Dirac Kernel there remains no such thing like acausal & superluminal propagation of initially precise position localized free single Dirac particle in the Foldy-Wouthuysen representation in contradiction to the principles of the theory of relativity.

CALCULATION:

“According to the basic principles of quantum mechanics, one defines a Hilbert space for each quantum-mechanical system. To be precise, this Hilbert space, suitable for quantum mechanics, is a linear vector space, usually of infinite dimensions and complex such that all infinite series occurring in it are convergent. Every measurable quantity or “observer” (e.g., energy, momentum etc.) has to be represented by self-adjoint

operator in the so defined Hilbert space. The state of the system at initial time t_0 is given by a vector $\Psi(t_0)$, often called “wave-function” in the so defined Hilbert space and is represented in this Hilbert space as” [23]

$$\Psi(t_0) = \sum_{k=1}^{\infty} c_k \Psi_k \text{_____} (1e)$$

where c_k are the components of the vector $\Psi(t_0)$ & are often called as ‘**expansion coefficients**’ of the wave-function $\Psi(t_0)$ and unit vectors Ψ_k are called ‘**basis vectors**’ or complete set of vectors of the Hilbert space.

The summation over the values of the variable ‘k’ in the above expression becomes an integral if the value of the variable ‘k’ lies in a continuum. We assume that ‘**basis vectors**’ Ψ_k satisfies the orthonormality property and accordingly, $\Psi(t_0)$ is **normalized** i.e., multiplied by a constant such that the integration of the scalar product $|\Psi(t_0)|^2$ over the entire volume space yields unity. At subsequent time t, the state of the system is given by

$$\Psi(t) = [\exp. (-i/\hbar) (Ht)] \Psi(t_0) \text{_____} (1f)$$

where i is equal to $[(-1)^{1/2}]$, \hbar is Planck’s constant having value equal to $\{[6.625 \times (10)^{-34}]/2\pi\}$ Joule-second and the self-adjoint operator H (“Hamiltonian”) represents the energy of the system.

“According to the Stone’s theorem [24], $\Psi(t)$ is the unique strong solution of the Cauchy problem” [23]

$$(i \hbar \partial \Psi / \partial t) = H \Psi(t)$$

Let us now take the unique case of a particular quantum-mechanical system that comprises a free dirac electron and venture out to define Hilbert space for the same. To begin with the aforesaid venture, we substitute, in the right hand side of the above mentioned Cauchy problem $[(i\hbar \partial \Psi / \partial t) = H \Psi(t)]$, the Dirac Hamiltonian operator that operates upon the wave-function $\Psi(t)$ and is given explicitly by the matrix-valued differential expression $[(H) = (c\alpha_x)(-i \hbar \partial / \partial x) + (c\alpha_y)(-i \hbar \partial / \partial y) + (c\alpha_z)(-i\hbar \partial / \partial z) + (\beta m_0 c^2)]$ & get

$$(i\hbar \partial \Psi / \partial t) = (c\alpha_x)(-i\hbar \partial \Psi / \partial x) + (c\alpha_y)(-i\hbar \partial \Psi / \partial y) + (c\alpha_z)(-i\hbar \partial \Psi / \partial z) + (\beta m_0 c^2) \Psi \text{_____} (1)$$

Where m_0 is rest mass of the electron,

c is light speed in vacuum,

$\alpha_x, \alpha_y, \alpha_z$ & β are Dirac matrices operators satisfying the relations

$$\{[\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1] \& [\alpha_x \alpha_y + \alpha_y \alpha_x = \alpha_x \alpha_z + \alpha_z \alpha_x =$$

$$\alpha_z \alpha_y + \alpha_y \alpha_z = \alpha_x \beta + \beta \alpha_x = \alpha_y \beta + \beta \alpha_y = \alpha_z \beta + \beta \alpha_z = 0]\}$$

It can be easily inferred from the above relations, satisfied by the dirac matrices $\alpha_x, \alpha_y, \alpha_z$ & β , that the Equation (1) above ought to be a set of partial differential equations upon which we elaborate at a little later stage.

Substituting the axial components of the linear momentum operator of the free dirac electron viz. $[p_x = (-i\hbar\partial/\partial x)]$, $[p_y = (-i\hbar\partial/\partial y)]$ & $[p_z = (-i\hbar\partial/\partial z)]$ in the above equation, we get

$$(i\hbar\partial\Psi/\partial t) = (p_x)(c\alpha_x)\Psi + (p_y)(c\alpha_y)\Psi + (p_z)(c\alpha_z)\Psi + (\beta m_0 c^2)\Psi \quad (1a)$$

Since, the Equation (1a), as already pointed out earlier, is a set of partial differential equations, so, the eigenvalues of the axial components of the linear momentum operator viz. $[p_x = (-i\hbar\partial/\partial x)]$, $[p_y = (-i\hbar\partial/\partial y)]$ & $[p_z = (-i\hbar\partial/\partial z)]$ also becomes variables in the above Equation (1a). As such, the Equation (1a) has got three variables comprising position, time and eigenvalues p of the linear momentum operator of the free dirac electron. Accordingly, using separation of the variable techniques, we find that the solutions of the above set of partial differential equations (1a) are, in-fact, an infinite number of the plane waves such that each one is of the form $\{[\exp.(-i/\hbar)(Ht)].\varphi(p).$ $[\exp.(i/\hbar)(\mathbf{p}\cdot\mathbf{r})]\}$. The superposition of these plane waves yields the following square-integrable wave packet that is also the solution of the above partial differential equation (1a).

$$\Psi(t) = A \int \varphi(p). [\exp.(-i/\hbar)(Ht)]. \exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\} d^3p \quad (1b)$$

where $A = [1/(2\pi\hbar)]^3$ is the normalization constant.

The perusal of the above equation (1b) and subsequent, comparison of the same with the above Equation (1e) above reveals that the normalized state vector $\Psi(t)$, for the case-in-hand, is defined in an infinite dimensional Hilbert space and, as per aforesaid convention and current literature practice [25], is having $\varphi(p)$ as ‘**expansion coefficients**’ and $[\exp.(-i/\hbar)(Ht)].[\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}]$ as ‘**basis vectors**’ in the aforesaid infinite dimensional Hilbert space. As such, conventionally and as per aforesaid current literature practice [25], the changes in the state vector $\Psi(t)$ with time have to be attributed to the corresponding changes in the time-dependent ‘**basis vectors**’ $[\exp.(-i/\hbar)(Ht)].[\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}]$ & the ‘**expansion coefficients**’ $\varphi(p)$ are then time-independent in the aforesaid infinite dimensional Hilbert space.

But, to the contrary, we shall now demonstrate that the spatial terms in the Equation (1) can be easily integrated out, as follows, in order to arrive at the obvious conclusion that the temporal term $[\exp.(-i/\hbar)(Ht)]$ in the wave-function $\Psi(t)$ of equation (1b) needs to be associated with the ‘**expansion coefficients**’ as $\{\varphi(p,t) = \varphi(p).[\exp.(-i/\hbar)(Ht)]\}$ and not with the ‘**basis vectors**’ as $[\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}]$.

Substituting the above equation (1b) in the right hand side of the equation (1), we get

$$\begin{aligned} i\hbar[\partial\Psi(t)/\partial t] &= A \int [(-i\hbar\partial/\partial x)(c\alpha_x) + (-i\hbar\partial/\partial y)(c\alpha_y) + (i\hbar\partial/\partial z)(c\alpha_z) + (\beta m_0 c^2)] [\varphi(p,t) \exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}] d^3p \\ i\hbar[\partial\Psi(t)/\partial t] &= A \int \varphi(p,t) [(c\alpha_x)(-i\hbar\partial/\partial x) + (c\alpha_y)(-i\hbar\partial/\partial y) + (c\alpha_z)(i\hbar\partial/\partial z) + (\beta m_0 c^2)] [\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}] d^3p \\ i\hbar[\partial\Psi(t)/\partial t] &= A \int \varphi(p,t) [(p_x)(c\alpha_x) + (p_y)(c\alpha_y) + (p_z)(c\alpha_z) + (\beta m_0 c^2)] [\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}] d^3p \\ i\hbar[\partial\Psi(t)/\partial t] &= A \int c[(\alpha_x)\{p_x\varphi(p,t)\} + (\alpha_y)\{p_y\varphi(p,t)\} + (\alpha_z)\{p_z\varphi(p,t)\} + (\beta m_0 c)\varphi(p,t)] [\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}] d^3p \end{aligned}$$

Multiplying both sides of the above equation by the special term $[\exp.(i/\hbar)(-\mathbf{p}_0\cdot\mathbf{r})]$, we get $[\exp.(i/\hbar)(-\mathbf{p}_0\cdot\mathbf{r})]i\hbar[\partial\Psi(t)/\partial t] = A \int [\exp.(i/\hbar)(-\mathbf{p}_0\cdot\mathbf{r})] c [(\alpha_x)\{p_x\varphi(p,t)\} + (\alpha_y)\{p_y\varphi(p,t)\} + (\alpha_z)\{p_z\varphi(p,t)\} + (\beta m_0 c)\varphi(p,t)] [\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}] d^3p$

Integrating both sides of the above equation over the entire co-ordinate space, we get $i\hbar[\partial]\{[\exp.(i/\hbar)(-\mathbf{p}_0\cdot\mathbf{r})]\Psi(t)\}d^3r/\partial t = A \int \int [\exp.(i/\hbar)(-\mathbf{p}_0\cdot\mathbf{r})] c [(\alpha_x)\{p_x\varphi(p,t)\} + (\alpha_y)\{p_y\varphi(p,t)\} + (\alpha_z)\{p_z\varphi(p,t)\} + (\beta m_0 c)\varphi(p,t)] [\exp.\{(i/\hbar)(\mathbf{p}\cdot\mathbf{r})\}] d^3p$

$$\varphi(\mathbf{p}, t) + (\beta m_0 c) \varphi(\mathbf{p}, t) [\exp.\{(i/\hbar)(\mathbf{p} \cdot \mathbf{r})\}] d^3 p \cdot d^3 r$$

Changing the order of integration and rearranging the terms in right hand side of the above equation, we get

$$i\hbar[\partial_t \{ [\exp.(i/\hbar)(-\mathbf{p}_0 \cdot \mathbf{r})] \Psi(t) \} d^3 r / \partial t] = A \int \{ [\exp.(i/\hbar)(-\mathbf{p}_0 \cdot \mathbf{r})] [\exp.(i/\hbar)(\mathbf{p} \cdot \mathbf{r})] d^3 r \} c [(\alpha_x) \{ p_x \varphi(\mathbf{p}, t) \} + (\alpha_y) \{ p_y \varphi(\mathbf{p}, t) \} + (\alpha_z) \{ p_z \varphi(\mathbf{p}, t) \} + (\beta m_0 c) \varphi(\mathbf{p}, t)] d^3 p$$

Further, evaluating the integrals in the right hand side of the above equation, we get

$$i\hbar[\partial_t \{ [\exp.(i/\hbar)(-\mathbf{p}_0 \cdot \mathbf{r})] \Psi(t) \} d^3 r / \partial t] = c [(\alpha_x) \{ p_{x0} \varphi(p_0, t) \} + (\alpha_y) \{ p_{y0} \varphi(p_0, t) \} + (\alpha_z) \{ p_{z0} \varphi(p_0, t) \} + (\beta m_0 c) \varphi(p_0, t)]$$

Now, substituting the equation (1b) in the left hand side of the above equation and further, evaluating the resulting integral in the left hand side of above equation, we get

$$i\hbar[\partial_t \varphi(p_0, t) / \partial t] = c [(\alpha_x) \{ p_{x0} \varphi(p_0, t) \} + (\alpha_y) \{ p_{y0} \varphi(p_0, t) \} + (\alpha_z) \{ p_{z0} \varphi(p_0, t) \} + (\beta m_0 c) \varphi(p_0, t)]$$

Since, no conditions have been imposed on the variable \mathbf{p}_0 and as such, we generalize the above equation by dropping the subscript 0 from the axial components of \mathbf{p}_0 and get

$$i\hbar[\partial_t \varphi(\mathbf{p}, t) / \partial t] = c [(\alpha_x) \{ p_x \varphi(\mathbf{p}, t) \} + (\alpha_y) \{ p_y \varphi(\mathbf{p}, t) \} + (\alpha_z) \{ p_z \varphi(\mathbf{p}, t) \} + (\beta m_0 c) \varphi(\mathbf{p}, t)] \quad \text{---(1c)}$$

The simple derivation of the above equation (1c), by first multiplying the aforesaid special term $[\exp.(i/\hbar)(-\mathbf{p}_0 \cdot \mathbf{r})]$ with the equation (1) and subsequently, integrating the latter, leads to the aforesaid obvious **conclusion** that the temporal term $[\exp.(-i/\hbar)(Ht)]$ in the wave-function $\Psi(t)$ of equation (1b) needs to be associated with the ‘**expansion coefficients**’ as $\{ \varphi(\mathbf{p}, t) = \varphi(\mathbf{p}) \cdot [\exp.(-i/\hbar)(Ht)] \}$ and not with the ‘**basis vectors**’ as $[\exp.\{(i/\hbar)(\mathbf{p} \cdot \mathbf{r})\}]$. With the inclusion of this abovementioned conclusion, the Equation (1b) can be rewritten as

$$\Psi(t) = A \int \varphi(\mathbf{p}, t) \cdot \exp.\{(i/\hbar)(\mathbf{p} \cdot \mathbf{r})\} d^3 p \quad \text{---(1d)}$$

“Let us now further develop and solve, as follows, the above Equation (1c) in the momentum space with an ulterior motive to have a glance at the consequences arising out of the aforesaid conclusion.

In the momentum space (p-space), the above equation (1c) can be rewritten as

$$i\hbar[\partial_t \varphi(\mathbf{p}, t) / \partial t] = (H\varphi) = (p_x \varphi)(c\alpha_x) + (p_y \varphi)(c\alpha_y) + (p_z \varphi)(c\alpha_z) + (\beta m_0 c^2) \varphi \quad \text{---(2)}$$

where $\varphi = \varphi(\mathbf{p}, t)$, $[(H) = (p_x)(c\alpha_x) + (p_y)(c\alpha_y) + (p_z)(c\alpha_z) + (\beta m_0 c^2)]$ and p_x, p_y & p_z are the eigenvalues of the respective axial components p_x, p_y & p_z of the linear momentum operator that is defined as multiplication operator in momentum space.

For solving the above equation (2), we consider the special case called Dirac Steady or stationary state of definite energy. This special case is of great practical importance because it is easiest to see what happens to $\varphi(\mathbf{p}, t)$ in the momentum space, when Dirac

particle moves with the passage of time, by studying the equation (2) above. For this we try a special solution of the form $\varphi(p,t) = f(t) \phi(p)$, a function of time 't' only multiplied by a function of momentum 'p' only. Substitution in the Equation (2) gives us the relation

$$\begin{aligned} &(-\hbar/i)f'(t) \phi(p) = Hf(t) \phi(p) = f(t) H\phi(p) \\ \text{or} \quad &(-\hbar/i)(f'/f) = (1/\phi)(H\phi) \end{aligned} \quad (3)$$

The left hand side of this equation (3) does not depend upon p, whereas the right hand side of this equation is independent of t. If they are always equal, neither side can depend upon either variable t or p. That is, each side must be a constant. Let us call this constant E. Then, $f' = -(i/\hbar) E f$, or $f = C e^{-(i/\hbar)Et}$ with an arbitrary constant factor C. Thus, the special solution is of the form

$$\varphi = e^{-(i/\hbar)Et} \phi(p) \quad \text{where } \phi \text{ satisfies } H\phi = E\phi \text{ and } C \text{ is included in } \phi \quad (3a)$$

Therefore, when the wave function is of this special form, the state has a definite energy E. For each value of definite energy 'E', a different particular solution ϕ [a solution of equation $H\phi = E\phi$] must be sought. Since, because of earlier supposed normalized wave-function Ψ , our functions ϕ_n corresponding to the set of energy levels E_n are not only orthogonal but also normalized in the momentum space, then, as a matter of fact, it can be shown that if all possible values of E_n {in-fact, the spectrum of E is purely absolutely continuous and is given by $(-\infty, -m_0c^2] \cup [+m_0c^2, +\infty)$ } and corresponding functions ϕ_n are worked out, any solution $\varphi(p)$ of Equation (2) can be written, at $t = 0$, as a linear combination of these special solutions of definite energy i.e." [21]

$$\varphi(p, t = 0) = \varphi(p) = \sum_{n=1}^{\infty} a_n \phi_n(p) \quad \text{where we have } a_n = \int_{-\infty}^{+\infty} \phi_n^*(p_0) \cdot \varphi(p_0) d^3p_0 \quad (3b)$$

Thus, at $t = 0$, we have the identity

$$\varphi(p) = \sum_{n=1}^{\infty} \phi_n(p) \int_{-\infty}^{+\infty} \phi_n^*(p_0) \cdot \varphi(p_0) dp_0 = \int_{-\infty}^{+\infty} \left[\sum_{n=1}^{\infty} \phi_n^*(p_0) \cdot \phi_n(p) \right] \varphi(p_0) d^3p_0 \quad (4)$$

Now, in order to further examine the extreme right hand side of the Equation (4) above, we need to digress a little for determining the nature of the wave-function ϕ . Towards this end of the said digression, we proceed from Equation (3) as follows:

The right hand side of equation (3), when equated to the constant E, becomes

$$\begin{aligned} &(1/\phi)(H\phi) = E \\ \text{or} \quad &E\phi = (H\phi) \end{aligned}$$

When the Dirac Hamiltonian Operator $\{ H = [(p_x)(c\alpha_x) + (p_y)(c\alpha_y) + (p_z)(c\alpha_z) + (\beta m_0 c^2)] \}$ is substituted in the above equation, we get

$$(E\phi) = (c\alpha_x)(p_x\phi) + (c\alpha_y)(p_y\phi) + (c\alpha_z)(p_z\phi) + (\beta m_0 c^2)\phi \quad (5)$$

Now, the momentum wave-function ϕ in the above Equation (5) needs to satisfy the following two conditions:

1. In momentum space the dynamical variables of the free Dirac electron are represented by operators which act on the momentum wave-function ϕ to yield their respective constant eigenvalues.

2. Since, in the above equation (5) the terms α_x , α_y , α_z & β are Dirac (4x4) matrix operators that satisfies the following relations

3.

$$\{[\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1] \text{ and } [\alpha_x\alpha_y + \alpha_y\alpha_x = \alpha_x\alpha_z + \alpha_z\alpha_x = \alpha_z\alpha_y + \alpha_y\alpha_z = \alpha_x\beta + \beta\alpha_x = \alpha_y\beta + \beta\alpha_y = \alpha_z\beta + \beta\alpha_z = 0]\},$$

so, ϕ , as a Dirac Operand in the equation (5) above, needs to be a column matrix that is having four components and is usually written as transpose of row matrix viz.

$$\{\phi = [\phi^1, \phi^2, \phi^3, \phi^4]^T\}.$$

This implies that as per aforesaid first condition, if we assign **constant** eigenvalues x , y & z respectively to the following components of the position operator of the free Dirac electron in the momentum space

$$[x = (i\hbar\partial/\partial p_x)], [y = (i\hbar\partial/\partial p_y)] \text{ \& } [z = (i\hbar\partial/\partial p_z)],$$

then, by solving the following position operator partial differential equations

$$\{[x\phi = (i\hbar\partial\phi/\partial p_x)], [y\phi = (i\hbar\partial\phi/\partial p_y)] \text{ \& } [z\phi = (i\hbar\partial\phi/\partial p_z)]\},$$

we conclude that, as per aforesaid second condition, each one of all the four components of the momentum wave-function ϕ , in the abstract p-space, ought to be written as

$$\phi^j(\mathbf{p}) = u^j \exp.\{(-i/\hbar)[(p_x)(x) + (p_y)(y) + (p_z)(z)]\} = u^j \exp.\{(-i/\hbar)(\mathbf{p}.\mathbf{r})\}$$

(where the superscript j varies from 1 to 4 and u^j is number)

Now, one needs to find out the exact nature of the number u^j in the above form of the four components of the momentum wave-function ϕ . Towards this goal, substituting

$$\{\phi = [\phi^1, \phi^2, \phi^3, \phi^4]\},$$

$$\text{where } \phi^j(\mathbf{p}) = u^j \exp.\{(-i/\hbar)[(p_x)(x) + (p_y)(y) + (p_z)(z)]\} = u^j \exp.\{(-i/\hbar)(\mathbf{p}.\mathbf{r})\},$$

in Equation (5), yields:

(a) two linearly independent solutions for $E_+ = +[p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m_0^2 c^4]^{1/2}$ viz.

$$u^1 = 1, \quad u^2 = 0, \quad u^3 = [c p_z / (E_+ + m_0 c^2)], \quad u^4 = [c (p_x + i p_y) / (E_+ + m_0 c^2)]$$

$$u^1 = 0, \quad u^2 = 1, \quad u^3 = [c (p_x - i p_y) / (E_+ + m_0 c^2)], \quad u^4 = [-c p_z / (E_+ + m_0 c^2)]$$

(b) two linearly independent solutions for $E. = -[p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m_0^2 c^4]^{1/2}$ viz.
 $u^3 = 1, \quad u^4 = 0, \quad u^1 = [c p_z / (E. - m_0 c^2)], \quad u^2 = [c (p_x + i p_y) / (E. - m_0 c^2)]$
 $u^3 = 0, \quad u^4 = 1, \quad u^1 = [c (p_x - i p_y) / (E. - m_0 c^2)], \quad u^2 = [-c p_z / (E. - m_0 c^2)]$

Now, after finding out the exact nature of the complex u^j values, as above, the only task left towards complete determination of wave-function ϕ is its normalization. As such, for the purpose of dirac-delta normalization of ϕ in the abstract momentum space, each u^j value of all the four above linearly independent solutions needs to be divided by the corresponding dirac-delta normalization constant viz.

$$[|u^1|^2 + |u^2|^2 + |u^3|^2 + |u^4|^2]^{1/2}$$

(where the notation $|u^j|$ denotes, here, the modulus of the complex u^j values).

The normalized u^j values so obtained after the aforesaid division are given by

$$\{u^j / (|u^1|^2 + |u^2|^2 + |u^3|^2 + |u^4|^2)^{1/2}\}$$

Thus, after the inclusion of the above normalized u^j values, the normalized components of the wave-function ϕ becomes

$$[\phi^j(\mathbf{p}) = \{u^j / (|u^1|^2 + |u^2|^2 + |u^3|^2 + |u^4|^2)^{1/2}\} \exp.\{(-i/\hbar)(\mathbf{p}.\mathbf{r})\}]$$

where j is having values 1 to 4],

This is the end of our aforesaid digression and now, armed with complete knowledge of the nature of the wave-function ϕ , we return to our main topic of further examining the extreme right hand side of the Equation (4) as follows:

Now, with the substitution of the above determined normalized components

$$[\phi^j(\mathbf{p}) = \{u^j / (|u^1|^2 + |u^2|^2 + |u^3|^2 + |u^4|^2)^{1/2}\} \exp.\{(-i/\hbar)(\mathbf{p}.\mathbf{r})\}] \text{ _____ (6)}$$

of the wave-function ϕ in extreme right hand side of the Equation (4), the following term

$$\sum_{n=1}^{\infty} \phi_n(\mathbf{p}).\phi_n^* (p_0) \text{ _____ (7)}$$

becomes dirac-delta function, that is defined in momentum space, and the Equation (4) becomes definition of this dirac-delta function.

Now, at this stage the only remaining term, that sans the complete knowledge of the Equation (4), is the index 'n' of the summation in the above Equation (7). As such, for having complete understanding of this index 'n', we proceed as follows: Substituting the Equation (6) in the Equation (3a), we get

$$\phi = e^{(-i/\hbar)Et} U \exp.\{(-i/\hbar)(\mathbf{p}.\mathbf{r})\} \text{ _____ (8)}$$

where $U = \{u^j / (|u^1|^2 + |u^2|^2 + |u^3|^2 + |u^4|^2)^{1/2}\}$ & j varies from 1 to 4.

Since, as already demonstrated above in the Equation (3b), the momentum wave-function φ is, in-fact, a linear combination of such special solutions, as are defined above in Equation (8), of definite energy, so, one can rewrite the Equation (8) above by using the Equation (3b) at a subsequent time instant 't', i.e.

$$\varphi = \varphi(\mathbf{p}, t) = \sum_{n=1}^{\infty} a_n e^{(-i/\hbar)E_n t} U_n \exp.\{(-i/\hbar)(\mathbf{p} \cdot \mathbf{r})\}$$

Changing the notation of the linear momentum operator from \mathbf{p} to \mathbf{p}' in the above Equation, we get

$$\varphi = \varphi(\mathbf{p}', t) = \sum_{n=1}^{\infty} a_n e^{(-i/\hbar)E_n t} U_n \exp.\{(-i/\hbar)(\mathbf{p}' \cdot \mathbf{r})\} \quad \text{_____ (9)}$$

Further, substituting the above Equation (9), in the Equation (1d), we get

$$\Psi(t) = A \int \left[\sum_{n=1}^{\infty} a_n e^{(-i/\hbar)E_n t} U_n \exp.\{(-i/\hbar)(\mathbf{p}' \cdot \mathbf{r})\} \right] \cdot \exp.\{(i/\hbar)(\mathbf{p}' \cdot \mathbf{r})\} d^3 p' \quad \text{_____ (10)}$$

where $U_n = \{u^j / (|u^1|^2 + |u^2|^2 + |u^3|^2 + |u^4|^2)^{1/2}\}$ & j varies from 1 to 4.

The essential reason why four components in the above Equation (10) are in general necessary to describe a state of positive or negative energy in the representation of Dirac theory corresponding to Equation (1) is that the Hamiltonian in this Equation (1) contains odd operators, specially the components $\alpha_x, \alpha_y, \alpha_z$ of the operator $\boldsymbol{\alpha}$. By performing Foldy-Wouthuysen (FW) transformation

$$\exp.(iS) = \beta [\beta m_0 c^2 + \boldsymbol{\alpha} \cdot \mathbf{p} \cdot \mathbf{c} + \beta E_n] / [2E_n(E_n + m_0 c^2)]^{1/2}$$

[where S is an Hermitian operator]

on the Equation (1), the latter becomes free of aforesaid odd operators and the positive and negative energy states in the FW representation are represented by wave-function $\Psi(t)_{(FW)}$ that is having only two components in each case, the other pair of components being identically zero. As such, the wave-function $\Psi(t)_{(FW)}$ for either positive or negative energy state can be easily derived from above Equation (10) in the following manner i.e.,

$$\Psi(t)_{(FW)} = \exp.(iS) \cdot \Psi(t) = A \int \left[\sum_{n=1}^{\infty} a_{n(FW)} e^{(-i/\hbar)E_n t} \exp.\{(-i/\hbar)(\mathbf{p}' \cdot \mathbf{r})\} \right] \cdot \exp.\{(i/\hbar)(\mathbf{p}' \cdot \mathbf{r})\} d^3 p'$$

Interchanging the order of Integration and summation in the above equation, we get

$$\Psi(t)_{(FW)} = \exp.(iS) \cdot \Psi(t) = A \sum_{n=1}^{\infty} \int a_{n(FW)} e^{(-i/\hbar)E_n t} \exp.\{(-i/\hbar)(\mathbf{p}' \cdot \mathbf{r})\} \cdot \exp.\{(i/\hbar)(\mathbf{p}' \cdot \mathbf{r})\} d^3 p' \quad \text{_____ (11)}$$

Since, the terms $a_{n(\text{FW})}$ and $[e^{(-i/\hbar)E_n t}]$ in the above Equation (11) do not depend upon the variable p , so, accordingly taking them out of the integral sign in the above Equation (11) yields the following equation;

$$\Psi(t)_{(\text{FW})} = \sum_{n=1}^{\infty} a_{n(\text{FW})} e^{(-i/\hbar)E_n t} \delta^3(\mathbf{r} - \mathbf{r}) \text{ for } t \geq 0 \quad \text{_____} \quad (12)$$

$$= 0 \quad \text{for } t < 0$$

The occurrence of the dirac-delta functions $\delta^3(\mathbf{r} - \mathbf{r})$ in the above Equation (12) shows that the position \mathbf{r} of the dirac free particle does not change with time. However, the phase of the position co-ordinate wave-function changes continuously with time in accordance with the factor $e^{(-i/\hbar)E_n t}$. As such, we consider this temporally constant position vector \mathbf{r} as an index 'n' in the Equation (4) above, conclude from the Fourier Transformation that $\Psi(t)_{(\text{FW})}$ is equal to $a_{n(\text{FW})}$ at time instant 't = 0' and note the orthogonality in momentum space. That is, as long as $\mathbf{r} \neq \mathbf{r}'$,

$$\int_{\mathbf{p}} [\phi_{\mathbf{r}}^* \cdot \phi_{\mathbf{r}'}] d^3p = 0 \quad \text{even if} \quad E_{\mathbf{r}} = E_{\mathbf{r}'}$$

At first sight, it looks very peculiar for the free Dirac particle to have its constant position vector \mathbf{r} as an index 'n' for its definite energy E in the Equation (4). In order to comprehend this peculiarity, the Dirac particle in question ought to be considered as “**free**” if and only if it is in an “**empty**” space where the probability wave-function $\Psi(t)$, as defined in the Equation (1b), exists as such for it. We can explain this by following the description of an “**empty**” space at page 111 of the book [44] as follows:

What we think of as “**empty**” space cannot be completely empty because that would mean that all the fields, such as probability quantum field $\Psi(t)$ of the free Dirac particle, would have to be exactly zero. In other words, generally we cannot just say that the ground state of the empty space is the state with no quanta - we have to solve the proper quantum field equations, with proper boundary conditions, and determine what the state of the field is. Such a state may or may not contain quanta. In particular, whenever the space has a boundary, the ground state of the field does contain quanta - this fact is called the vacuum polarization effect. In QED, this is a very well known, and experimentally verified effect. For example, two conducting parallel plates attract each other, even if they are not charged and placed in otherwise empty space (this is called the Casimir effect). One can understand this attraction very easily. Namely, the vacuum fluctuations of the probability field of the dirac electrons, trapped inside the conducting plates, may create in an empty space virtual electron-positron pairs. These charged particles induce virtual polarization charges in the conducting plates (it means virtual photons are created, travel to plates, and reflect from them). Hence, the plates become virtually charged, and attract one another during a short time when the existence of the virtual charges, and virtual photons, is allowed by the Heisenberg principle. All in all, a net attractive force between plates appears.

Moreover, the value of the probability field $\Psi(t)$ and its rate of change with time are like the position and velocity of the particle: the uncertainly principle implies that more accurately one knows one of these quantities, the lesser accurately one

can know the other. So, in the “empty” space the probability field $\Psi(t)$ cannot be exactly zero, because then it would have both a precise value (zero) and a precise temporal rate of change (also zero). There must be minimum amount of uncertainty or quantum fluctuations in the value of the field probability $\Psi(t)$. One can think of these quantum fluctuations as pairs of the Dirac particles (particle and antiparticle) that appear together at some time, move apart, and then come closer again and annihilate each other. These particles are the virtual particles that cannot be observed directly with the particle detector.

The constant E of the Equation (3) represents energy of the system described by the free Dirac equation. Since, the spectrum of the free Dirac Hamiltonian operator has a negative part, this system can be in a state with negative energy. Moreover, the unboundedness of the free Dirac Hamiltonian operator provides us with an infinite energy reservoir. Originally, our intention was to describe a single free Dirac particle, for which the occurrence of the negative energies E_- , as above, is most peculiar fact. In the light of the facts explained in the foregoing Para, we can interpret the occurrence of these negative energies in the spectrum of the free unbounded multi-particle Dirac Hamiltonian operator having infinite energy reservoir, as theoretical prediction of the existence of the virtual pairs of the Dirac particle and its antiparticle at any point in the “empty” space surrounding the real particle and accordingly, a real free Dirac particle, that is understood to be one as being described by the free Dirac equation in an “empty” space, must have its energy state E distinct from all virtual pairs of the Dirac particle and anti-particle at other points of the said “empty” space in accordance with the Pauli Exclusion Principle. This distinction in the value of the aforesaid energy E is possible if and only if the free Dirac particle is understood to have its constant position vector \mathbf{r} as an index ‘n’ for its definite energy E in the Equation (4).

With the aforesaid consideration of the constant position vector \mathbf{r} as an index ‘n’, the Foldy-Wouthuysen (FW) transformation of the Equation (4) needs to be rewritten as

$$\varphi(\mathbf{p})_{FW} = \int_{-\infty}^{+\infty} \sum_{\mathbf{r}} \{ [\phi_{\mathbf{r}}^*(\mathbf{p}_0) \cdot \phi_{\mathbf{r}}(\mathbf{p})] \exp.(iS) \} \varphi(\mathbf{p}_0) d^3\mathbf{p}_0 \quad (13)$$

where S depends upon \mathbf{p}_0 .

Since, \mathbf{r} 's are distributed over a continuum in co-ordinate space, the sum over the “indices” \mathbf{r} in the above Equation (13) is really equivalent to a Dirac-delta function integral over the values of \mathbf{r} , namely

$$\varphi(\mathbf{p})_{FW} = \int_{-\infty}^{+\infty} \{ (1/2\pi)^3 \int [\phi_{\mathbf{r}}^*(\mathbf{p}_0) \cdot \phi_{\mathbf{r}}(\mathbf{p})] \exp.(iS) d^3\mathbf{r} \} \varphi(\mathbf{p}_0) d^3\mathbf{p}_0 \quad (14)$$

Since, the Foldy-Wouthuysen (FW) transformation $\exp.(iS)$ is independent of the position \mathbf{r} and depends solely upon \mathbf{p}_0 , so, the above Equation (14) can be rewritten as

$$\varphi(\mathbf{p})_{FW} = \int_{-\infty}^{+\infty} \{ (1/2\pi)^3 \int [\phi_{\mathbf{r}}^*(\mathbf{p}_0) \cdot \phi_{\mathbf{r}}(\mathbf{p})] d^3\mathbf{r} \} \exp.(iS) \varphi(\mathbf{p}_0) d^3\mathbf{p}_0 \quad (15)$$

The perusal of the above Equation (15) reveals that even after the Foldy-Wouthuysen (FW) transformation, the Dirac-delta function obtained in Equation (7) remains in-tact in its momentum space dependence and the transformation of $\varphi(\mathbf{p})_{FW}$ is still a point transformation in the FW representation. This revelation is quite unlike the one that was made by L. L. Foldy and S. A. Wouthuysen in their classic paper [19]; wherein a coordinate space wave-function which in the old representation corresponded to a state in which the particle was definitely located at one point, passes over in the new FW representation into a wave-function which apparently correspond to the particle being spread out over a finite region.

Further, the Dirac-delta function $\{(1/2\pi)^3 \int [\phi_r^*(\mathbf{p}_0) \cdot \phi_r(\mathbf{p})] d^3\mathbf{r}\}$ in the above Equation (15) is nothing but Dirac propagator or Kernel at $t = 0$. In Dirac steady states of definite energy, this kernel K cannot depend upon the absolute time but instead is a function of time interval $[(t_2 - t_1) = (t - 0) = t]$ only. As a consequence of this fact, the wave function $\varphi(\mathbf{p})_{FW}$ in the Equation (15) varies periodically with passage of time t and, using Equation (3a), can be written as

$$\varphi(\mathbf{p},t)_{FW} = \int_{-\infty}^{+\infty} \{(1/2\pi)^3 \int [\phi_r^*(\mathbf{p}_0) \cdot \phi_r(\mathbf{p})] \exp[-(i/\hbar)E_r t] d^3\mathbf{r}\} \exp.(iS) \varphi(\mathbf{p}_0) d^3\mathbf{p}_0 \quad (16)$$

Therefore, from the above Equation (16), we conclude that the Dirac free particle Kernel $K(\mathbf{p},t;\mathbf{p}_0,0)$ at time instant t can be written as

$$K(\mathbf{p},t;\mathbf{p}_0,0) = (1/2\pi)^3 \int \{\phi_r^*(\mathbf{p}_0) \cdot \phi_r(\mathbf{p}) \exp[-(i/\hbar) E_r t]\} d^3\mathbf{r}$$

$$K(\mathbf{p},t;\mathbf{p}_0,0) = (1/2\pi)^3 \int k(\mathbf{p},t;\mathbf{p}_0,0) d^3\mathbf{r} \quad \text{where } [k(\mathbf{p},t;\mathbf{p}_0,0) = \{\phi_r^*(\mathbf{p}_0) \cdot \phi_r(\mathbf{p}) \exp[-(i/\hbar) E_r t]\}] \quad (17)$$

Now, as in [22], we interpret the integrand in the above Equation (17) of Dirac free particle Kernel K at time instant t in terms of the solution ϕ_r of a constant Hamiltonian that is defined in the Equation (3a). We notice first that $\phi_r(\mathbf{p})$ is the amplitude that if we are at position \mathbf{r} , we measure the linear momentum \mathbf{p} , while on the other hand $\phi_r^*(\mathbf{p}_0)$ is the amplitude that if we measure linear momentum \mathbf{p}_0 , we are at position \mathbf{r} . Also, the term $\{\exp[-(i/\hbar)E_r t]\}$ is the amplitude to be at the position \mathbf{r} at time t if we are already at position \mathbf{r} at the time $t = 0$. Thus, in the above Equation (17), the total amplitude $[k(\mathbf{p},t;\mathbf{p}_0,0) = \{\phi_r^*(\mathbf{p}_0) \cdot \phi_r(\mathbf{p}) \exp[-(i/\hbar)E_r t]\}]$ to get from linear momentum \mathbf{p}_0 at time $t = 0$ to linear momentum \mathbf{p} at time t is integrated over the alternatives that are to be divided into various positions spread over a neighborhood of dimensions of the order of a Compton wave-length of the Dirac particle about the point \mathbf{r} in the co-ordinate space.

Now, let us divide the time interval $[t,0]$ into N infinitesimally small parts $\Delta t_k \equiv [t_k, t_{k-1}]$ of equal length $\epsilon = [(t - 0)/N]$, where $k = 1,2,3 \dots N$, $t_0 \equiv 0$, $t_N \equiv t$. Basic trajectories in the phase space are straight line segments in the 3D – configuration momentum space with pieces $\Delta \mathbf{p}_{(k)} \equiv \mathbf{p}_{(k)} - \mathbf{p}_{(k-1)}$, where we define $\mathbf{p}_{(0)} \equiv \mathbf{p}_0$, $\mathbf{p}_{(N)} \equiv \mathbf{p}$. The position vector in any time interval Δt_k has a constant value $\mathbf{r}_{(k)}$. As such, the standard procedure of the iterations leads to the following representation of the Equation (17):

$$\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_k \mathbf{r}_{N-1}$$

$$K(p,t;p_0,0) = (1/2\pi)^{3(N-1)} \int \dots \int k(p,t;p_{(N-1)},t-\epsilon) \cdot k(p_{(N-1)},t-\epsilon;p_{(N-2)},t-2\epsilon) \dots k(p_k,k\epsilon;p_{(k-1)},(k-1)\epsilon) \dots k(p_1,\epsilon;p_0,0) d^3r_1 d^3r_2 \dots d^3r_k \dots d^3r_{N-1} \quad (18)$$

When we take the limit $[N \rightarrow \infty]$ or $[\epsilon \rightarrow 0]$ in the above Equation (18), the constant position vectors r_1, r_2, \dots, r_{N-1} are represented by adjacent points on a path that is connecting the starting point r_0 at time instant $(t_0 - \epsilon = -\epsilon)$ to the end point $(r_N = r)$ at time instant $(t_N + \epsilon = t + \epsilon)$ in the co-ordinate space. Since, the integrands in the right hand side of the Equation (18) are to be integrated over a neighborhood of dimensions of the order of a Compton wave-length of the Dirac particle about these adjacent points r_1, r_2, \dots, r_{N-1} respectively, in the co-ordinate space, so, in the limit $[N \rightarrow \infty]$ these integrations yield Feynman integral $K(r,t;r_0,0)$ over all the paths connecting the starting point r_0 at $t = 0$ to the end point $(r_N = r)$ at $t = t$ in the co-ordinate space. This implies that in the limit $[N \rightarrow \infty]$ the right hand side of the Equation (18) is equal to the Feynman path integral $K(r,t;r_0,0)$ in the co-ordinate space i.e.,

$$K(r,t;r_0,0) = \lim_{N \rightarrow \infty} K(p,N\epsilon;p_0,0) \quad (19)$$

A mere casual look at the above Equations (18) and (19) is sufficient enough to reveal the truth that the Dirac Kernel $K(r,t;r_0,0)$ in the co-ordinate space is, in fact, composed, in the time sequence, of the product of N number of the Dirac Kernels

$$\{M = (1/2\pi)^3 \int_{r_k} k(p_k,k\epsilon;p_{(k-1)},(k-1)\epsilon) d^3r_k\},$$

each of which sequentially yields integration over the alternatives $r_{(k)}$ on the path between r_0 and $(r_N = r)$ during the passage of time. As such, in the limit $[N \rightarrow \infty]$ or $[\epsilon \rightarrow 0]$ these infinite number of the Dirac Kernels M must retain their individual identities and at any intermittent time instant 't = kε' the corresponding individually retained Dirac Kernel M involves integration over a neighborhood of dimensions of the order of a Compton wave-length of the Dirac particle about the position r_k .

DISCUSSION:

The key to understanding this aforementioned preceding section's rather unusual last statement, that the individual identities of the new Dirac Kernels M above are to be retained in the limit $[N \rightarrow \infty]$ or $[\epsilon \rightarrow 0]$ and at any intermittent time instant 't = kε' any one of the above corresponding individually retained Dirac Kernel M involves integration over a neighborhood of dimensions of the order of a Compton wave-length of the Dirac particle about the position r_k , lies in the fact that in the Foldy-Wouthuysen (FW) representation $\phi(p,t)_{FW}$, the operator representative for the position x (for the sake convenience only x-component has been chosen but the results obtained are equally applicable to other components also) of the free Dirac particle is no longer the operator

$$[x = (i\hbar\partial/\partial p_x)];$$

but, the rather complicated operator:

$$x_{FW} = \exp.(iS) x \exp.(-iS)$$

where $\exp.(iS) = \beta[\beta m_0 c^2 + \boldsymbol{\alpha} \cdot \mathbf{p} \cdot c + \beta E_r] / [2E_r(E_r + m_0 c^2)]^{1/2}$

Or $x_{FW} = \exp.(iS) (i\hbar\partial/\partial p_x) \exp.(-iS)$
Or $x_{FW} = x - (i\beta\alpha c/2E_r) + [\{i\beta c^2(\boldsymbol{\alpha} \cdot \mathbf{p})\mathbf{p} - (1/2i)pc^2(\boldsymbol{\alpha} X \boldsymbol{\alpha}) X \mathbf{p}\} / \{2E_r(E_r + m_0 c^2)p\}]$

As such, at any intermittent time instant ‘ $t = k\varepsilon$ ’ the above complicated position operator x_{FW} in-fact reminds one of the contributions, depending on $k(p_k, k\varepsilon: p_{(k-1)}, (k-1)\varepsilon)$, to the corresponding individually retained Dirac Kernel M over a neighborhood of dimensions of the order of a Compton wave-length of the Dirac particle about the constant eigenvalue r_k . This foregoing statement coupled with the fact, that in the Equation (12), the position operator eigenvalue r is to be taken as the constant of motion only during the infinitesimal time-period ε , entails one to define this position operator eigenvalue r as a new position operator [$x'_{FW} = (i\hbar\partial/\partial p_x)$] called ‘mean position operator’ in the limit [$N \rightarrow \infty$] or [$\varepsilon \rightarrow 0$] in the Foldy-Wouthuysen (FW) representation such that its operator-representative in old representation is given by:

$$\begin{aligned} x' &= \exp.(-iS) x'_{FW} \exp.(iS) \\ &= \exp.(-iS) (i\hbar\partial/\partial p_x) \exp.(iS) \\ &= x'_{FW} + (i\beta\alpha c/2E_r) - [\{i\beta c^2(\boldsymbol{\alpha} \cdot \mathbf{p})\mathbf{p} + (1/2i)pc^2(\boldsymbol{\alpha} X \boldsymbol{\alpha}) X \mathbf{p}\} / \{2E_r(E_r + m_0 c^2)p\}] \end{aligned}$$

As already stated above, this newly defined ‘mean position operator x'_{FW} or x' ’ is to be taken as the constant of motion only during the infinitesimal time-period ε and accordingly, in the limit [$N \rightarrow \infty$] or [$\varepsilon \rightarrow 0$] its temporal rate of change must corresponds to the “**conventional**” velocity operator. Strictly speaking, one exactly finds the same result in [19] when one takes the time derivative of this newly defined ‘mean position operator x'_{FW} or x' ’ in the Foldy-Wouthuysen (FW) representation as well as in the old representation.

CONCLUSIONS:

In [19], the authors have cited that the left hand side of the Equation (21) therein, is not like Dirac-delta function in its space dependence, because of the occurrence of Foldy-Wouthuysen (FW) transformation $\exp.(iS)$ in its right hand side. This lead them to the conclusion that Foldy-Wouthuysen (FW) transformation $\exp.(iS)$ results in the physical spread out of the earlier point-like Dirac particle over a finite region. In [23] at page 18, it is this physical spread that is shown to be responsible for the acausal or superluminal propagation of initially localized free Dirac particles in the Foldy-Wouthuysen representation.

Contrary to the foregoing Para conclusions, the perusal of the Equation (15) above and its interpretation in the succeeding lines reveal that Foldy-Wouthuysen (FW) transformation $\exp.(iS)$ can be taken out of the dirac-delta function of the dirac particle in momentum space and accordingly, the Dirac particle has been taken as a point particle for all purpose in the subsequent calculations.

Further, in the foregoing discussion, we have shown that the complicated position operator x_{FW} of the Dirac particle, in the Foldy-Wouthuysen representation,

needs not be taken as indicative of the physical spread of the earlier point-like Dirac particle over a finite region, as is the case in [19], rather needs to be taken as the reminder of the contributions, depending on $k(p_k, k\varepsilon; p_{(k-1)}, (k-1)\varepsilon)$, to the corresponding individually retained Dirac Kernel M over a neighborhood of dimensions of the order of a Compton wave-length of the Dirac particle about the constant eigenvalue r_k .

Lastly, the time evolution operator [$\exp. (-i/\hbar) (Ht)$] of the Equation (1f) above is a path-integral operator of the Equation (19) in the Hilbert space of the wave-function $\Psi(t)$ of Equation (1b) above. The free time evolution of this path-integral operator in the Equation (19) is in the form of a time-sequenced “product” of an infinite series of individual Dirac Kernels M . For the time-duration ε , each of this Dirac Kernel M involves integration over a neighborhood of dimensions of the order of a Compton wave-length of the Dirac particle about x'_{FW} or x' (It is because of this integration only that x'_{FW} or x' has been given the name ‘mean position operator’) to be taken as constant only for this time-duration ε such that in the limit [$N \rightarrow \infty$] or [$\varepsilon \rightarrow 0$] the temporal rate of change of x'_{FW} or x' yields conventional velocity operator. Consequently, in the free time evolution of the said path-integral operator in the Equation (19), there remains no such thing like the acausal or superluminal propagation [23] of initially precise position localized free Dirac particles in the Foldy-Wouthuysen representation.

REFERENCES:

- [1] A. J. Bracken, Flohr and G. F. Molloy, *Time-evolution of highly localized positive-energy states of the free Dirac electron*, Proc. Roy. Soc. London A461, 3633 – 3645 (2005).
- [2] Iwo Bialynicki-Birula, *Weyl, Dirac, and Maxwell equations on a lattice as unitary cellular automata*, Phys. Rev. D 49, 6920–6927 (1994) [Issue 12 – 15 June 1994]
- [3] [A. J. Bracken](#), [G. F. Melloy](#), J.Phys. A32, 6127-6139, (1999)
- [4] P.A.M. Dirac, *The Principles of Quantum Mechanics, 4th Ed.* (Clarendon Press, Oxford, 1958).
- [5] E. Schrödinger, Sitzungsber. Preuss. Akad. Wiss. Phys-Math. Kl. XXIV, 418 (1930); III, 63 (1931).
- [6] L. Landau and R. Peierls, Z. Phys. 69, 56 (1931), transl. in: D. ter Haar (Ed.), *Collected papers of L.D. Landau* (Pergamon, Oxford, 1965).
- [7] M.H.L. Pryce, Proc. Roy. Soc. London 150A, 166 (1935); 195A, 62 (1948).
- [8] A. Papapetrou, Praktika Acad. Ath´enes 15, 404 (1940); C. Møller, Commun. Dublin Inst. Adv. Stud. A, No. 5 (1949).
- [9] T.D. Newton and E.P. Wigner, Rev. Mod. Phys. 21, 400 (1949).
- [10] A.S. Wightman, Rev. Mod. Phys. 34, 845 (1962).

- [11] S. Tani, *Prog. Theoret. Phys.* 6, 267 (1951); D.G. Currie, T.F. Jordan and E.C.G. Sudarshan, *Rev. Mod. Phys.* 35, 350 (1963); T.F. Jordan and N. Mukunda, *Phys. Rev.* 132, 1842 (1963); W. Weidlich and A.K. Mitra, *Nuovo Cim.* (10) 30, 385 (1963); T.O. Philips, *Phys. Rev.* 136 B893 (1964); G.N. Fleming, *Phys. Rev.* 137, B188 (1965); *J. Math. Phys.* 7, 1959 (1966); R.A. Berg, *J. Math. Phys.* 6, 34 (1965); A. Galindo, *Nuovo Cim.* (10) 37, 413 (1965); A.J. K'alnay, "The Localization Problem," in M. Bunge (Ed.) *Studies in the Foundations, Methodology and Philosophy of Science*. Vol. 4, *Problems in the Foundations of Physics* (Springer Verlag, Berlin, 1971); A.J. K'alnay and P.L. Torres, *Phys. Rev. D*7, 1707 (1973); G.C. Hegerfeldt *Phys. Rev. D*10 3320 (1974); Z. Haba, *Nuovo Cim.* A(11)32, 174 (1976); B.K. Skagerstam, *Int. J. Theoret. Phys.* 15, 213 (1976); J.F. Perez and I.F. Wilde, *Phys. Rev. D*16, 315 (1977); E.P. Wigner and R.F. O'Connell, *Phys. Lett.* 61A, 353 (1977); *Phys. Lett.* 67A 319 (1978) L.L. Foldy and S.A. Wouthuysen, *Phys. Rev.* 78, 29 (1950).
- [12] H. Bacry, *J. Math. Phys.* 5, 109-111 (1964).
- [13] A.O. Barut and S. Malin, *Rev. Mod. Phys.* 40, 632-650 (1968).
- [14] G.C. Hegerfeldt and S.N.M. Ruijsenaars, *Phys. Rev. D*22, 377 (1980); G.C. Hegerfeldt, *Phys. Rev. Lett.* 54, 2395 (1985); *Nucl. Phys. B Proc. Suppl.* 6, 231 (1989).
- [15] J.M. Knight, *J. Math. Phys.* 2, 459 (1964); A.L. Licht, *J. Math. Phys.* 4, 1443 (1963); R. Haag, *Local Quantum Physics* (Springer-Verlag, New York, 1992).
- [16] H. Bacry, *Localizability and Space in Quantum Physics* (Springer-Verlag, Berlin, 1988); V.V. Dodonov and S.S. Mizrahi, *Phys. Letts* 177A, 394 (1993); M-T. Jaekel and S. Reynaud, *Phys. Lett.* A220, 10 (1996); R. Omnes, *J. Math. Phys.* 38, 708 (1997); S.T. Ali, *Int. J. Theoret. Phys.* 37, 365 (1998).
- [17] O. Klein, *Die Reflexion von Elektronen an einem Potentialsprung nach der Relativischen Dynamik von Dirac*, *Z. Phys.* **53** (1929) 157.
- [18] B. R. Holstein, *Klein's paradox*, *Am. J. Phys.* **66** (1998) 507.
- [19] L.L Foldy, S.A Wouthuysen, *Phys. Rev.*78, 29 (1950)
- [20] R. P. Feynman, *Rev. Mod. Phys.* 20, 267 (1948). Unpublished notes by Feynman on path integrals for the Dirac equation in one dimension are described in S. S. Schweber, *Rev. Mod. Phys.* 58, 449 (1986).
- [21] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, New York, 1965).
- [22] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, p. 117 (McGraw-Hill, New York, 1965).

- [23] B. Thaller, *The Dirac Equation*, Text and Monographs in Physics (Springer-Verlag, Heidelberg 1992)
- [24] M. Reed and B. Simon: *Methods of Modern Mathematical Physics I, Functional Analysis*, Academic Press, New York, San Francisco, London 1972, 2nd edition 1980 – Notes 1.2. 2.3
- [25] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, p. 102 (McGraw-Hill, New York, 1965).
- [26] G. V. Riazanov, *Zh. Eksp. Theor. Fiz.* 33, 1437 (1957) [*Sov. Phys. J.E.T.P.* 6, 1107 (1958)].
- [27] T. Nakamura, *J. Math. Phys.* 32, 457 (1991)
- [28] D. G. C. McKeon and G. N. Ord, *Phys. Rev. Lett.* 69, 3 (1992).
- [29] G. J. Papadopoulos and J. T. Devreese, *Phys. Rev. D* 13, 2227 (1976).
- [30] G. Rosen, *Phys. Rev. A* 28, 1139 (1983).
- [31] M. A. Kayed and A. Inomata, *Phys. Rev. Lett.* 53, 107 (1984).
- [32] B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman, *Phys. Rev. Lett.* 53, 419 (1984).
- [33] A. O. Barut and I. H. Duru, *Phys. Rev. Lett.* 53, 2355 (1984).
- [34] T. Ichinose and H. Tamura, *J. Math. Phys.* 25 , 1810 (1984).
- [35] A. M. Polyakov, in *Fields, Strings, and Critical Phenomena, Les Houches XLIX*, edited by E. Brézin and J. Zinn-Justin, (North-Holland, Amsterdam, 1990).
- [36] J. Ambjørn, B. Durhuus, and T. Jonsson, *Nucl. Phys. B* 330 , 509 (1989).
- [37] L. S. Schulman, *Phys. Rev.* 176 , 1558 (1968).
- [38] T. Jacobson, *J. Phys. A* 17 , 2433 (1984).
- [39] T. Jacobson, *Phys. Lett. B* 216 , 150 (1989).
- [40] F. A. Berezin and M. S. Marinov, *Ann. Phys. (NY)* 104 , 332 (1977).
- [41] F. Ravndal, *Phys. Rev. D* 21, 2823 (1980).
- [42] A. T. Ogielski and J. Sobczyk, *J. Math. Phys.* 22 , 2060 (1981).
- [43] B. Gaveau and L. S. Schulman, *Phys. Rev. D* 36 , 1135 (1987).

[44] Stephen Hawking, *A Brief History of Time*, Bantam Press Edition 1988,
Transworld Publishers, Great Britain