SUMMARY

LIGHT CHARACTERISTICS Vs. SINGULARITY IN CLASSICAL GENERAL RELATIVITY

I. INTRODUCTION

It is well known that in one of the famous **exact** solutions [Karl Schwarzschild space time Metric] of Field Equations of General Relativity a singularity exists at *Schwarzschild radius.* In past different ADHOC coordinate systems have been proposed to get over this problem e.g.

i. Eddington and Finkelstein

ii. Kruskal - Szekeres

In this research paper a different method (See para **II** below) is discussed (without using ADHOC coordinate approach) that helps in formulating a new invariant space-time metric which is free from singularity as mentioned above. This in turn **avoids** at times peculiar intrepretations e.g.

If an object collapses to *Schwarzschild radius* it continues to collapse to zero radius of infinite density and further, light cannot escape from this type of object.

Or Emergence of event horizon which would shield the object in this state from view.

II. BRIEF METHODOLOGY

Based on fundamental principles of Physics, light characteristics are determined **first**. The equations thus obtained leads to formulation of **new** invariant space-time metric.

This approach however still satisfies the following principles and established methodology of Physics :

i. In all inertial reference frames speed of Light in vaccum is maximum andsame - Special Theory of Relativity(S.T.R.)

ii. Equivalence principle of General Relativity(G.T.R.)

iii. Principle Idea of G.T.R. is maintained i.e. matter influences the space time around it. Thus the Eucludiean geometry of empty space is changed to non - Euclidiean(Curved space time) when massive body is introduced in it. Hence the motion of test particle (including photon) takes place along an invariant geodesic pertaining to this curved space time. However in my method test partcile mass is explicitly cancelled while describing its motion (see eqn. (5) & (6) of main manuscript) which is different from G.T.R. (where it is implicitly assumed to have been cancelled by using equivalence principle during formulation)

iv. No preferred reference frame should be used when describing a law of Physics.

v. Correspondence Principle i.e., this new space time metric under approximation reduces to Karl Schwarzschild metric and further reduces to Flat Space Time Metric asymtotically.

III. PACS CLASSIFICATION (2008) under which this research paper falls

04.	...	General Relativity and Gravitation
04.20.-q	...	Classical General Relativity
04.20.Jb	...	Exact Solutions
04.70.Bw	...	Classical Black Holes
04.20.Dw	...	Singularity and Cosmic Censorship

LIGHT CHARACTERISTICS Vs. SINGULARITY IN CLASSICAL GENERAL RELATIVITY

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Using a different approach an invariant curved space time metric in Classical General Relativity is obtained which is free from Singularity at *Schwarzschild radius*

Nomenclature:

G = Gravitational Constant
h = Planck's Constant
◊ = Origin, an arbitrary point located in an otherwise **empty** space time.
M = Gravitating mass (assuming it to be the only one compact mass which exists in an otherwise empty space time). It is spherically symmetric, nonrotating w.r.t. origin ◊ and at rest at the origin ◊.
r = Distance [1] from origin ◊ where mass M is located. (Note : it is not radial distance or radius. However it becomes radial only where $\varphi = 0$) -------- see eqn (8)
φ = GM/r = Gravitational Potential due to mass M ... see eqn (6)
K = Refrence frame located where $\varphi = 0$ and at rest w.r.t. origin ◊
r_g = Distance where $r = 2GM/c_\infty^2$ *(also called Schwarzschild radius)*
m_0 = Test particle rest mass w.r.t. refrence frame K, where $\varphi = 0$. Note $m_0 << M$
t, t_0 = time measured by clocks stationary at their respective locations.
v = Equivalent Velocity of test particle w.r.t. ref. frame K -------- see eqn (6)
R = Distance r from origin ◊ to the surface of symmetrical mass M
c_∞ = Velocity of propagation of light / photon in Vaccum where $\varphi = 0$
c_r = Velocity of propagation of light / photon in Vaccum at distance r where $\varphi \neq 0$
$\underline{D}$ = By definition

Subscripts:

$_0$ = w.r.t. reference frame K
$_r$ = distance w.r.t. origin ◊
$*$ = Properties at r_g

Abbreviations :

G.T.R. = General Theory of Relativity,
S.T.R. = Special Theory of Relativity
w.r.t. = with respect to
eqn. = equation

Superscripts : [a,b]-(foot notes on page 3)

1. Characteristics of Light / Photon in a Gravity field :

Let a photon of energy hf_∞ , moving with velocity c_∞ , in vaccum, exist in refrence frame K. During its motion, let this photon come into the region (vaccum) of curved space time due to mass M[a], then its velocity will start decreasing as :

$$c_r = c_\infty / (1 + GM/rc_\infty^{2})^2 \qquad \text{...............(1)}$$

or approximately

$$c_r \cong c_\infty (1 - 2GM/rc_\infty^{2}) \qquad \text{...............(2)}$$

Eqn(2) can be obtained by setting ds = 0 of eqn (19) which is based on G.T.R.

1.1 Derivation of equation (1) :

Brief Introduction :

In the accelerated motion of a reference frame inertial forces will always be **parallel** to one another. Thus it is imposible to find a non-inertial reference frame in which the inertial forces have the same direction and magnitude variations, at different locations, as in the gravitational field.

Then Principle of Equivalance (G.T.R.) holds only for small region of space, so small that within this region the gravitational field can be assumed to be uniform. Thus gravitational field is **not** apparent and all that is implied in G.T.R. is that properties of space time in grav field are analogous to those in non-inertial reference frame **locally.**

1.1.1 *Methodology :*

Let a test particle P of rest mass m_0 having negligible velocity exist in a reference frame *K* During its motion let it come into the region of curved space time due to mass M (>>m_0). The velocity of this test partcile will now start increasing.

The *free fall* motion of this test particle P is analysed, **indirectly**, as follows :
The change in potential energy of this *configuration* as the test particle is displaced from **rest** (w.r.t origin ◊) at distance *r*[a] to a rest position (w.r.t origin ◊) at infinity or **vice versa** $\underline{D}$ $\frac{GMm_0}{r}$ --- (3)

(see foot note *b* below.)

Note on Eqn(3) :

(i) Eqn. (3) is **time invariant** because it inherently asumes that when the test particle moves from one **Rest** position (w.r.t origin ◊) at distance *r* to another **Rest** position (w.r.t origin ◊) at infinity, its rest energy $m_0 c_\infty^{\ 2}$ remains constant during this process.

(ii) Gravitational mass and Inertial mass m_0[a] of test particle are same (as explained in para (i) above also) - Equivalence principle of General Relativity.

Some authors distinguish between Weak and Strong Equivalence principle, Weak refers to free fall motion of body whereas Strong refers to whole of physics. In this research paper this equivalence principle refers to Whole of Physics [1].

(iii) At infinity, the potential energy of this configuration taken as reference is zero.

(iv) In the present analysis it is sufficient that $r \geq R$. However this analysis can be extended to $r < R$ where *M* is to be replaced by function *M(r)* [4]

Let us now determine the "**Equivalent**" change in test particle's Kinetic Energy i.e., imagine at infinity, if this test particle which is intially at rest w.r.t. ◊ and also in ref. frame *K* is imparted energy equal in magnitude as given by eqn (3). Then this is uniquely given by in ref. frame *K* as(S.T.R.),

$$= \frac{m_0 c_\infty^{\ 2}}{\sqrt{1 - v^2 / c_\infty^{\ 2}}} - m_0 c_\infty^{\ 2} \qquad \text{---------} (4)$$

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[a] *For detail definitions of symbols i.e. M, r, K etc. - see page 2 - under para* **"Nomenclature"**

[b] *In the final exact solution of "field equations" a similar term GM/r appears and is interpreted as "Gravitational Potential" due to mass M [see eqn.(19) Appendix (A)]. However the exact definition of Potential Energy* [4] *should be used first w.r.t. total 'configuration of M and test particle mass* m_0 *' i.e., eqn. (3) rather than gravitational potential GM/r due to M alone.*

Where v is "**Equivalent**" velocity of test particle in reference frame *K*,
Repeating, as the phenomena expressed by eqn. (3) and eqn. (4) are independently time invariant by virtue of their respective definitions. So from eqn (3) & (4)

$$\frac{m_0 c_\infty^{\ 2}}{\sqrt{1-v^2/c_\infty^{\ 2}}} - m_0 c_\infty^{\ 2} = \frac{GMm_0}{r} \qquad \text{---------- (5)}$$

Rewriting eqn. (5) after cancelling m_0

$$\frac{1}{\sqrt{1-\frac{v^2}{c_\infty^{\ 2}}}} = 1 + \frac{GM}{rc_\infty^{\ 2}} = 1 + \frac{\varphi}{c_\infty^{\ 2}} \qquad \text{where } \varphi = GM/r \qquad \text{---------- (6)}$$

Eqn. (6) describes the free fall motion of test particle P.

1.1.2. Left side of Equation (6) is well known as "**Lorentz Factor**"in S.T.R. and Right Side is its equivalent factor derived for G.T.R.

Consider a case where, $\left(\frac{v^2}{c_\infty^{\ 2}} << 1\right)$ then eq: (6) becomes,

$$v^2 \cong 2\frac{GM}{r} = 2\varphi \qquad \text{---------- (7)}$$

Eqn. (7) is inherently present in Karl Schwarzschild's "space time metric"**. *(See eqn (19); Appendix A)***

Equation (6) & (7) show that "**dynamic condition**" involving equivalent velocity v *(where* $\varphi = 0$) of mass m_O can now be replaced by "**static condition**" at distance *r* by assigning potential $\varphi\ (r)$ due to mass $M(>>m_O)$ or **vice versa.**

1.1.3. Similarly at a distance where $\varphi = 0$, let a small element of length Δr_0 and small time interval Δt_0 (at same point) exist in reference frame *K*. Then at any other distance *r* where $\varphi \neq 0$, correspondingly , let Δr be length of same element and Δt be the time interval respectively,

Assuming further, corresponding to distance *r*, "equivalent" velocity v, as defned by eqn. (6) remains constant, while determining Δr and Δt, one gets :

Using eqn.(6)
under "static condition" at distance *r* :
where $\varphi \neq 0$

$$\Delta r = \Delta r_0 \Big/ \left[1 + \frac{\varphi}{c_\infty^{\ 2}}\right] \qquad \text{.................. (8)}$$

Note : Δr is an exact differential
Similarly :

Using eqn.(6)
under "static condition" at distance *r* :
where $\varphi \neq 0$

$$\Delta t = \Delta t_0 \left[1 + \frac{\varphi}{c_\infty^{\ 2}}\right] \qquad \text{.................. (9)}$$

1.1.4. In a ref. frame *K*, let the velocity of light / photon be c_∞ which is determined [3] uniquely by using ***stationary*** "length" & "time" standards (i.e. $\Delta r_0,\ \Delta t_0$) in ref. frame *K*. Using single clock refer to experimental methods of Fizeau / Focault.

Since small length and time intervals ($\Delta r,\ \Delta t$) where $\varphi \neq 0$, (eqn. 8, 9) are function of $\Delta r_0,\ \Delta t_0$ respectively, so they are also unique and well defined (even though they vary with *r*).

Similarly let c_r be the velocity of same photon under **free fall**, at a distance *r* where $\varphi \neq 0$

c_r is determined uniquely by using ***stationary*** "length" & "time" standards, (i.e. $\Delta r,\ \Delta t$) located at distance *r*.

Now one can relate c_r and c_∞ as

$$c_r / c_\infty = \left(\Delta r / \Delta r_0\right) / \left(\Delta t / \Delta t_0\right) \quad \text{.................. (10)}$$

By using eqn. **(8), (9)** which relates these stationary standards, eqn. (10) becomes

$$c_r = c_\infty / \left[1 + \frac{\varphi}{c_\infty^2}\right]^2 \quad \text{.....................(11)}$$

So the derivation / proof of eqn(1) is thus obtained

Approximately eqn(11) becomes

$$c_r \cong c_\infty \left(1 - \frac{2\varphi}{c_\infty^2}\right) \quad \text{.....................(12)}$$

Note : It is better to use eqn. (11) rather than approximate eqn.(12) to avoid misconception that $\boldsymbol{c_r \to 0 \ at \ r = \frac{2GM}{c_\infty^2}}$

1.2. Second method of proof (Variational Principle)

Light takes a path(null geodesic) such that spacetime interval $ds^2 = 0$ (see Appendix A)

$$ds^2 = 0 = c_\infty^2 \Delta t_0^2 - \Delta r_0^2 = c_r^2 \Delta t^2 - \Delta r^2 \quad \text{.....................(13)}$$

which gives by using approximate eqn.(7).

$$c_\infty^2 \left[\Delta t \sqrt{1 - \frac{2\varphi}{c_\infty^2}}\right]^2 = \left[\Delta r / \sqrt{1 - \frac{2\varphi}{c_\infty^2}}\right]^2 \quad \text{...................(14)}$$

$$\therefore c_r = \frac{\Delta r}{\Delta t} = c_\infty \left(1 - \frac{2\varphi}{c_\infty^2}\right) \quad \text{...................(15)}$$

Eqn.(15) again leads to $c_r \to 0$ at $r = 2GM/c_\infty^2$. This is due to use of approximate eqn.(7). However, if exact eqn. (6) is used, it will give correct interpretation as given by eqn. (11).

Eqn. (15) can also be derived by setting $ds^2 = 0$ *in Eqn. (19) of Appendix A*

2. Case studies :

As no singularity exists at $r = 2GM/c_\infty^2$, however following cases are examined.

(a) If a photon with velocity c_∞ (where $\varphi = 0$) falls to a distance r_g where $\varphi \neq 0$ then

At $r = r_g = 2GM/c_\infty^2$, let its velocity be c_*

Using eqn. (11) ; $c_* = \frac{4}{9} c_\infty = .44 c_\infty$

(b) At $r = r_g$, using eqn. (9)

$\Delta t = \Delta t_* = \frac{3}{2} \Delta t_0$ (*Karl Sch*; method gives $\Delta t = \infty$ at $r = r_g$)

(c) If a material particle falls from rest where $\varphi = 0$ to a distance at $r = r_g$, (where $\varphi \neq 0$) then using eqn. (6), its *equivalent* velocity $v = v_* = \frac{\sqrt{5}}{3} c_\infty = .745 c_\infty$

(Note $v_* < c_\infty$)

APPENDIX A

3. In this appendix method for obtaining the **invariant** space time metric corresponding to the gravitating mass *M* is discussed which is **different** from Field Equation approach of G.T.R.

Nomenclature:

x, y, z	=	Orthogonal carterian coordinates
r, θ, ψ	=	polar coordinates (Orthogonal)
dt	=	differential time interval where $\varphi \neq 0$
dr	=	differential length where $\varphi \neq 0$
γ	=	$1 - 2GM/rc_\infty^2$
λ	=	$1 / \left[1 + GM / rc_\infty^2\right]^2$
S.T.R	=	Special Theory of Relativity ; G.T.R. = General Theory of Relativity

Subscripts:

o	=	conditions where $\varphi = 0$.

3.1. In S.T.R. one uses cartesian type coordinates. The equation for line element in flat space time is

$$ds^2 = c_\infty^2 dt_o^2 - \left(dx_o^2 + dy_o^2 + dz_o^2\right) \quad \text{................. (16)}$$

where ds^2 is **invariant** w.r.t. inertial ref.frames moving at uniform velocity w.r.t. each other.
In "differential form" using Polar Coordinates Eqn (16) can be written as :

$$ds^2 = c_\infty^2 dt_o^2 - \left(dr_o^2 + r^2 d\theta_o^2 + r^2 \sin^2\theta \, d\psi_o^2\right) \quad \text{....................(17)}$$

3.2. A simple but general form of space time interval in G.T.R. is of the type :

$$ds^2 = g_{\mu\upsilon} dx^\mu dx^\upsilon \quad \text{.................... (18)}$$

One of the famous exact solution of Field equations of G.T.R. i.e. curved space time metric(also known as Karl Schwarzschild metric), in the vicinity of mass *M* is

$$ds^2 = \left(1 - \frac{2GM}{rc_\infty^2}\right) dt^2 c_\infty^2 - \left[\left(1 - \frac{2GM}{rc_\infty^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\psi^2\right)\right] \quad \text{...........(19)}$$

This equation has a **singularity** at a distance *r* given by $r = 2GM/c_\infty^2$, (also known as *Schwarzschild radius*) [5]

The law of motion is given by extremizing the integral i.e.,

$$\delta \int ds = 0 \quad \text{...............(20)}$$

3.3. Curved space time and Gravitation interaction are effects due to **existence** of matter (like two sides of same coin) other similar examples are Energy and Mass, Photon and Material particles. The definitions of "differential" **Length and Time intervals** obtained by eqns (8), (9), and the fact that length and time intervals do not change normal to the direction of motion one obtains by using eqn. (8)(9) & (17) the **invariant spacetime 'Metric' (differential form)** in the vicinity of mass *M* as :

$$ds^2 = c_\infty^2 \lambda dt^2 - \left[\frac{dr^2}{\lambda} + r^2 (d\theta^2 + \sin^2\theta \, d\psi^2) \right] \quad \text{............. (21)}$$

where $\lambda = 1/(1+GM/rc_\infty^2)^2$

Note: coefficients of dt^2 (i.e., λ) and coefficients of dr^2 (i.e., $1/\lambda$) have been obtained by using eqn (8), (9) & (6).

3.4. A corollary emerges from eqn. (8) & (9), under free fall, i.e.,

product space $*$ time = $(\Delta r) * (\Delta t) = (\Delta r_0) * (\Delta t_0) =$ Constant and Independent of distance r

3.5 Summerising the **Metric** defined by eqn (21) represents

(i) The grav. field which is static

(ii) Field is spherically symmetric

(iii) Metric becomes that of Flat space time as $r \to \infty$

(iv) For $(GM/rc_\infty^2) << 1$, we can approximate

$$1 / \left(1 + \frac{GM}{rc_\infty^2}\right)^2 \cong 1 - \frac{2GM}{rc_\infty^2}$$

i.e. $\lambda \to \gamma$ or Eqn (21) → Eqn (19)

(Note: for the Sun near its surface) $GM/rc_\infty^2 \cong 2x10^{-6}$

3.6 Having removed the singularity at $r = 2GM/c_\infty^2$, in the present analysis r=0 becomes an artificial singularity [w.rt. eqn. (8), (9), (11) & (21)] with a condition that as r→0; $M(r)/r$ → finite value or 0 [4], [5]

Conclusion :

(i) The singularity which exists at Schwarzschild *radius* $r = 2GM/c_\infty^2$ in eqn (19) is now removed. [2]

(ii) Light Frequency / Wavelength **shift**, deflection of its path and related phenomena under gravity particularly with respect to qursistellar objects (QSO) [2] can now be determined without ambiquity by using exact equation (11) & (21)

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