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Introduction Hyper sphere volume
And
Volume in n-dimensional hyper sphere

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Abstract :
Calculate the volume of hyper sphere in n-dimensional in Euclidian space and find the volume Vn in the n-dimensional where R is radius and using the geometry

Introduction:

The hyper sphere is generalization the circle in 2-shpere and sphere in 3D let n tuples point $(x_1, x_2, \ldots\ldots\ldots\ldots x_n)$ write the volume an n-dimensional hyper sphere S_r in point .

$$S_r = \{(x_1, x_2, x_3, \ldots\ldots\ldots x_i) \mid x_i \in \text{‘} \}$$
$$x_1^2 + x_2^2 + \ldots\ldots\ldots + x_i^2 = R^2$$

Where R is radius give the last $\sum_{i=1}^{n} x_i^2 = R^2$ in spherical case give the line segment extending distance R in each direction $V_1 = 2R$ the case of n=2 corresponding to a circle $V_2 = 2\pi R^2$, n=3 corresponding to sphere of volume $V_3 = \frac{4}{3}\pi R^3$
By using Cavalier's principle . we can write the calculated the volume of any close solid as

$$V_3 = \int_{z_i}^{z_j} A(z)dz$$

The analysis volume . we can find 1-spher

$$V_1(R) = \int_{-R}^{-R} dx = 2R$$

The volume in 2-Sphere

$$V_2(R) = \int_{-R}^{-R} 2\sqrt{R^2 - x^2}dx$$

By the definition integration

$$x = Rs\,\text{in}\,\theta \Rightarrow dx = Rc\,\text{os}\,\theta d\theta$$

$$V_2(R) = \int_0^{\pi} 2\sqrt{R^2 - R^2\sin^2\theta}R\,Cos\,\theta d\theta$$

$$V_2(R) = 2R^2\int_0^{\pi} Cos^2\theta d\theta = 2R^2\int_0^{\pi}\frac{1+Cos\,2\theta}{2}d\theta = \pi R^2$$

The volume in the 3-Sphere

$$V_3(R) = \int_{-R}^{R} V_2(\sqrt{R^2 - x^2})dx$$

$$V_3(R) = \int_{-R}^{R} \pi(R^2 - x^2)dx = \pi(2R^3 - \frac{1}{3}R^3) = \frac{4}{3}\pi R^3$$

The volume in the 4D

$$V_4(R) = \int_{-R}^{R} V_3(\sqrt{R^2 - x^2})dx$$

$$V_4(R) = \frac{4}{3}\pi\int_{-R}^{R}(R^2 - x^2)^{\frac{3}{2}}dx$$

$$x = Rs\,\text{in}\,\theta \Rightarrow dx = Rc\,\text{os}\,\theta d\theta$$

$$V_4(R) = \frac{4}{3}\pi R^4\int_0^{\pi} Cos\,\theta d\theta = \frac{4}{3}\pi R^4 . \frac{3\pi}{8}$$

$$V_4(R) = \frac{1}{2}\pi^2 R^4$$

The volume in the 5D

$$V_5(R) = \int_{-R}^{R} V_4(\sqrt{R^2 - x^2})dx$$

$$V_5(R)=\frac{1}{2}\pi^2\int_{-R}^{R}(R^2-x^2)^2dx=\frac{\pi^2}{2}.\frac{16}{15}R^5=\frac{18}{15}\pi^2R^5$$

The volume in the 6D :

$$V_6(R)=\int_{-R}^{R}V_5(\sqrt{R^2-x^2})dx$$

$$V_6(R)=\frac{8}{15}\pi^2\int_{-R}^{R}(R^2-x^2)^{\frac{5}{2}}dx$$

$$V_6(R)=\frac{8}{15}\pi^2.\frac{5\pi R^6}{16}=\frac{1}{6}\pi^3R^6$$

The volume in 7D

$$V_7(R)=\int_{-R}^{R}V_6(\sqrt{R^2-x^2})dx$$

$$V_7(R)=\frac{1}{6}\pi^2\int_{-R}^{R}(R^2-x^2)^3dx$$

$$V_7(R)=\frac{1}{6}\pi^3.\frac{32R^7}{35}=\frac{16}{105}\pi^3R^7$$

The volume in 8D

$$V_8(R)=\int_{-R}^{R}V_7(\sqrt{R^2-x^2})dx$$

$$V_8(R)=\frac{16}{105}\pi^3\int_{-R}^{R}(R^2-x^2)^{\frac{7}{2}}dx=\frac{\pi^4}{24}R^8$$

The volume in 9D

$$V_9(R)=\int_{-R}^{R}V_8(\sqrt{R^2-x^2})dx$$

$$V_9(R)=\frac{\pi^4}{24}\int_{-R}^{R}(R^2-x^2)^4dx=\frac{32}{945}\pi^4R^9$$

The volume in 10-D

$$V_{10}(R)=\int_{-R}^{R}V_9(\sqrt{R^2-x^2})dx$$

$$V_9(R) = \frac{32}{945}\int_{-R}^{R}(R^2 - x^2)^{\frac{9}{2}}dx = \frac{1}{120}\pi^5 R^{10}$$

<u>The table give 10D in space</u>

n	v_{n-1}	v_n
1	1	2R
2	2	πR
3	π	$\frac{4}{3}\pi R^3$
4	$\frac{4}{3}\pi$	$\frac{\pi^2}{2}R^4$
5	$\frac{\pi^2}{2}$	$\frac{8\pi^2}{15}R^5$
6	$\frac{8\pi^2}{15}$	$\frac{\pi^3}{6}R^6$
7	$\frac{\pi^3}{6}$	$\frac{16\pi^3}{105}R^7$
8	$\frac{16\pi^3}{105}$	$\frac{\pi^4}{24}R^8$
9	$\frac{\pi^4}{24}$	$\frac{32\pi^4}{945}R^9$
10	$\frac{32\pi^4}{945}$	$\frac{\pi^5}{120}R^{10}$

We Assume $n \geq 10$ what the volume

Suppose $V_{n-1} = S_{n-1}R^{n-1}$

$$V_n = \int_{-R}^{R} V_{n-1}(\sqrt{R^2 - x^2})dx$$

$$V_n = S_{n-1}\int_{-R}^{R}(R^2 - x^2)^{\frac{n-1}{2}}dx$$

Where S the surface area in n-1

We can write

$$V_n = \int_{-R}^{R} V_{n-1}(\sqrt{R^2 - x^2})dx$$

By the integer

$$V_n = \frac{S_n}{n} \int_{-R}^{R} (R^2 - x^2)^{\frac{n-1}{2}} dx$$

$$x = R\cos\theta \Rightarrow dx = -R\sin\theta d\theta$$

$$V_n = \frac{S_n}{n} R^n \int_0^{\pi} Sin^n \theta d\theta$$

$$V_{n-1} = \frac{S_n R^n}{n}$$

$$V_n = V_{n-1} \int_0^{\pi} Sin^n \theta d\theta$$

The beta function give by $B(m,n) = \int_0^1 x^{n-1}(1-x)^{m-1}dx$:

Now we prove that

$$\int_0^{\pi} Sin^n \theta d\theta = B(\frac{n+1}{2}, \frac{1}{2})$$

$$x = Sin\theta \Rightarrow dx = Cos\theta d\theta$$

$$Cos\theta = (1-x^2)^{\frac{1}{2}} \Rightarrow d\theta = (1-x^2)^{-\frac{1}{2}}dx$$

$$x^2 = y \Rightarrow 2x\,dx = dy \Rightarrow dx = \frac{1}{2} y^{-\frac{1}{2}} dy$$

$$\int_0^{\pi} Sin^n \theta d\theta = 2\int_0^1 x^n (1 - x^2)^{-\frac{1}{2}} dx$$

$$\int_0^{\pi} Sin^n \theta d\theta = \int_0^1 y^{\frac{n+1}{2}-1} (1 - y)^{\frac{1}{2}-1} dy$$

I will definition the relationship between gamma and beat function :

$$B(n,m) = \frac{\Gamma n \Gamma m}{\Gamma n + m}$$

Prove that give by the definition gamma function

$$\Gamma n \Gamma m = \int_0^{\alpha} x^{m-1} e^{-x} dx \int_0^{\alpha} y^{n-1} e^{-y} dy$$

$$\Gamma n \Gamma m = \int_0^{\alpha}\int_0^{\alpha} y^{n-1} x^{m-1} e^{-(x+y)} dx dy$$

$$x = a^2, y = b^2 \Rightarrow dx = 2ada, dy = 2bdb$$

$$\Gamma n \Gamma m = 4\int_0^{\alpha}\int_0^{\alpha} a^{2n-1} b^{2m-1} e^{-(a^2+b^2)} dadb$$

By change the polar coordination

$$\Gamma n \Gamma m = 4\int_0^{\alpha}\int_0^{\frac{\pi}{2}} r^{2n+2m-1} e^{-(r^2)} dr d\theta$$

$$\Gamma n \Gamma m = \int_0^{\alpha} z^{n+m-1} e^{-z} dz \int_0^{\frac{\pi}{2}} Cos^{2m-1}\theta Sin^{2n-1}\theta d\theta$$

$$\Gamma n \Gamma m = (\Gamma n + m)(B(n,m))$$

Now the integer give

$$I_n = \int_0^{\pi} Sin^n \theta d\theta = B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{n+1}{2})\Gamma\frac{1}{2}}{\Gamma(\frac{n}{2}+1)}$$

Now give the nice formula to find the n volume

Where

$$x_1^2 + x_2^2 + \ldots\ldots\ldots\ldots\ldots + x_n^2 = R^2$$

$$\int_0^\alpha \ldots \int_0^\alpha e^{-(x_1^2 + x_2^2 + \ldots\ldots\ldots\ldots\ldots + x_n^2)} dx_1 \ldots\ldots\ldots\ldots dx_n$$

$$V_n = (\int_0^\alpha e^{-x_1^2} dx_1)^n = (\Gamma \frac{1}{2})^n = (\sqrt{\pi})^n$$

$$S_n \int_0^\alpha r^{n-1} e^{-r^2} dr = \pi^{\frac{n}{2}}$$

$$S_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma \frac{n}{2}}$$

And n-V give

$$V_n = \frac{2\pi^{\frac{n}{2}}}{n \Gamma \frac{n}{2}} R^n$$

This nice formula to give the volume in n space