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Retarded and Advanced Potentials

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General note: Here we consider only conventional electrodynamics and not those where basic principles are changed (like Wheeler and Feynman electrodynamics). Also, it should be noted that under consideration here is a theory of electromagnetic field (Maxwell's equations); but not electrodynamics as a whole because the currents (sources) are given in the past and in the future. We do not ask how they were obtained. The conservation of energy or momentum is not valid here. The conservation laws would be valid if we included mechanics.

In every physical situation the electromagnetic field is unique, it is not just any solution of Maxwell's equations but the solution that satisfies some initial and boundary conditions. The consideration of these initial-boundary conditions will uniquely determine which formulas that include retarded and advanced potentials are to be used.

Let us write down all the formulas we need. If we measure time in cm and introduce potential A^k we have:

$$\Box A^k = -4\pi j^k, \quad A^k{}_{|k} = 0, \quad j^k{}_{|k} = 0 \tag{1}$$

Let us introduce a scalar field $\Phi=e_k A^k$, where e_k is a constant vector (it is a unite vector, any covariant derivative of which equals to zero. Four independent constant vectors exist in a flat (zero curvature) four-dimensional space). We have:

$$\Box\Phi = -4\pi\rho \tag{2}$$

where $\rho=e_k j^k$. The Green's formula is written for a scalar field and it is the reason why we reduced vector equation (1) to a scalar equation (2). Green's function for this equation is:

$$G^{(\pm)} = \frac{\delta\,(t'-t\pm|\vec{x}\,'-\vec{x}\,|)}{|\vec{x}\,'-\vec{x}\,|} \tag{3}$$

Where +/- indicates retarded and advanced Green's functions.

Suppose we have a point P in the space with coordinates t,x,y,z. The sources of the field ρ (t,x,y,z) are given in the past and in the future in the whole space. Suppose the initial data are given at the hyperplane t=0. The field Φ which corresponds to given initial and boundary conditions is also unique in the whole space in the past and in the future. Let us apply a Green's formula to the 4-dimensional volume which is bounded by the hyperplane $t_1=t-\lambda$, the hyperplane $t_2=t+\lambda$, and some remote side hypersurface which is a sphere of radius λ+>λ during the time from t_1 to t_2. Here λ is an arbitrary positive constant (notice that we can apply a Green's formula to any part of the space - not just to the hyperplane with initial data where λ=t, t1=0). Making a proper insulation for the point P and applying Green's formula we have:

$$\Phi(t,\vec{x}) = \Phi_{vr}(t,\vec{x}) + \Phi_{sr}(t,\vec{x}) \tag{4}$$

$$\Phi(t,\vec{x}) = \Phi_{va}(t,\vec{x}) + \Phi_{sa}(t,\vec{x}) \tag{5}$$

where:

$$\Phi_{vr}(t,\vec{x}) = \int\limits_{r'\le\lambda} \rho\,(t-r',\vec{x}\,')r'dr'd\Omega' \tag{6}$$

$$\Phi_{sr}(t,\vec{x}) = \frac{1}{4\pi}\int\limits_{\substack{t'=t-\lambda\\ r'=\lambda}} (\Phi + r'\frac{\partial\Phi}{\partial r'} + r'\frac{\partial\Phi}{\partial t'})d\Omega' \tag{6'}$$

$$\Phi_{va}(t,\vec{x}) = \int\limits_{r'\le\lambda} \rho\,(t+r',\vec{x}\,')r'dr'd\Omega' \tag{7}$$

$$\Phi_{sa}(t,\vec{x}) = \frac{1}{4\pi} \int\limits_{\substack{t'=t+\lambda \\ r'=\lambda}} (\Phi + r'\frac{\partial\Phi}{\partial r'} - r'\frac{\partial\Phi}{\partial t'})d\Omega' \qquad (7')$$

In the equations above a spherical coordinates centered at the point P were introduced where: $r' = |\vec{x}' - \vec{x}|$, $d\Omega' = \sin(\theta)d\theta\, d\varphi$. The value of $\lambda+$ ensures that integral over the side surface disappears (it is out of reach of delta functions). $\Phi_{vr}(t,x,y,z)$ is a retarded volume potential given by (6). The integration goes over the ball of the radius λ centered at the point P. $\Phi_{sr}(t,x,y,z)$ is a retarded sphere potential given by (6'). It is taken over the sphere of the radius λ centered at P at the retarded time $t-\lambda$. (7) and (7') are advanced volume and sphere potentials. The formulas (4) and (5) express the physical field $\Phi(t,x,y,z)$ through retarded or advanced volume and sphere potentials. This physical field reflects a unique physical situation and remains the same no matter which formula (4) or (5) is used. Retarded or advanced volume and sphere potentials have no physical meaning by themselves.

We can prescribe arbitrary initial data say at $t=0$ and use (4) to find the field in the future ($t>0$) and use (5) to find the field in the past ($t<0$). Please notice a very important moment: in the theory of electromagnetic field we can not cut off the past and create any initial data we like to impose at $t=0$. Actually, giving the initial data we are responsible not only for the future but also for the past. We can not do here what we used to do in dynamics of material point - set off a canon and create initial position and velocity of the shell. Giving the initial data without correlation with the past usually puts us in the trouble with the past infinity (some unacceptable requirements on radiation or field the sources of which reside at infinity in the past. See below).

Applying the formulas (4) and (5) in some physical situation we are not restricted in choosing the value of λ. Suppose the point of observation $P(t,x,y,z)$ is chosen outside the region where the sources are. If we choose a small λ and apply formulas (4) and (5) then the volume potentials will be zero since the volume of integration does not reach the sources. The value of the field will be given only by the sphere potentials (retarded or advanced. We consider that the physical situation is unique and the field $\Phi(t,x,y,z)$ is known in all the space). If we increase λ then at some point all the sources will be included in the volume of integration. The field $\Phi(t,x,y,z)$ will be given by the sum of the volume and the sphere potentials. Notice, that if we further increase λ then the volume potentials won't change because no more sources will appear in the volume of integration. That means that the sphere potentials also won't depend on λ in spite of the fact that the sphere of integration will be increasing (its radius is λ) and the integration will be performed over the increasingly remote regions of space. It is quite a remarkable quality of a sphere potentials. Let us introduce the special notations:

$$\Phi_{in}(t,\vec{x}) \equiv \Phi_{sr}(t,\vec{x})_{\lambda\to\infty}; \quad \Phi_{out}(t,\vec{x}) \equiv \Phi_{sa}(t,\vec{x})_{\lambda\to\infty}$$

where λ is big enough to include all the sources inside the sphere of integration. These are the incoming and outgoing radiation which are connected to the point of observation P. It will be the same whether you integrate over the closest sphere that include all the sources inside it or over the infinite sphere. The important notice: while Φ_{sr} and Φ_{sa} as well as Φ_{vr} and Φ_{va} have no physical meaning by themselves (they are mathematical tools - integrals taken over the physical field Φ) the incoming Φ_{in} and outgoing Φ_{out} radiation do have a kind of physical meaning which we will discuss later.

Let us make some example calculations to clarify our point. Suppose a positive charge e (e=1 for simplicity) rests at the point $x=y=0$, $z=-a$, $a>0$, which means that we have a source: $\rho=\delta(x)\delta(y)\delta(z+a)$. This charge produces potential $\Phi=1/R$ where

$R=\mathrm{sqrt}[x^2+y^2+(z+a)^2]=\mathrm{sqrt}[r^2+a^2+2ra\cos(\theta)]$.

Let us choose the observation point at $x=y=z=0$ where $\Phi=1/a$ and see how the formulas (4) and (5) work in this simplest physical situation. We have:

$$\Phi=\int_{-\lambda}^{\lambda}\frac{\delta(z+a)dz}{z}+\frac{1}{2}\int_{0}^{\pi}\Big[\frac{1}{R}+r\frac{\partial}{\partial r}\Big(\frac{1}{R}\Big)\Big]_{r=\lambda}\sin(\theta)\,d\theta \tag{8}$$

where in volume potential we integrated over x and y , and in sphere potential we integrated over φ. We have: $\frac{\partial}{\partial r}\left(\frac{1}{R}\right)=-\frac{1}{R^3}\left(a\cos(\theta)+r\right)$. Now, we want to replace the integration over θ by the integration over R. We have: RdR=-ar.sin(θ)dθ. When θ changes from π to 0, R changes from |λ-a| to |λ+a|. We have:

$$\Phi=\int_{-\lambda}^{\lambda}\frac{\delta(z+a)dz}{z}+\frac{1}{4a\lambda}\int_{|\lambda-a|}^{\lambda+a}\Big[1+\frac{a^2-\lambda^2}{R^2}\Big]dR=$$
$$\int_{-\lambda}^{\lambda}\frac{\delta(z+a)dz}{z}+\frac{1}{4a\lambda}\Big[R+\frac{a^2-\lambda^2}{R}\Big]_{|\lambda-a|}^{\lambda+a} \tag{9}$$

If λ<a then the volume potential is zero and the sphere potential is 1/a. If λ>a then the volume potential is 1/a and the sphere potential is zero. The sphere potential will be zero if λ goes to infinity. That means that incoming and outgoing radiation which is connected to the point of observation are zero. We see on this example that formulas (4) and (5) really work fine at any λ. If there are no currents (sources) in the space, then from (4) and (5) we can conclude that $\Phi_{out}(t,x,y,z)=\Phi(t,x,y,z)=\Phi_{in}(t,x,y,z)$. In this case incoming radiation equally turns to outgoing radiation and they are coincide with the real physical field. For example: if we have some infinite plain wave $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ then: $\Phi_{in}(t,x,y,z)=\Phi_{out}(t,x,y,z)=e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ which can be checked by integration.

Let us take another example of a vacuum field. $\Phi_+=(1/r)e^{i(r+t)}$ is an incoming (to the origin of coordinates r=0) wave. $\Phi_{-}=(1/r)e^{i(r-t)}$ is an outgoing wave. Both of them have singularity at r=0 which means that we can not apply the formulas (4) and (5) to this fields. But let us try and see what happens. For the origin of coordinates (r=0) applying integral Φ_{in} to Φ_+ we get $2ike^{ikt}$ which can be written symbolically as: $\Phi_{in}(\Phi_+)=2ike^{ikt}$. Outgoing radiation integral taken on incoming wave is zero as well as incoming radiation integral taken on outgoing wave. Outgoing radiation integral taken on outgoing wave is not zero but its value 2ik (at t=0) is not right (we should expect infinity). This gives us some idea how these integrals work. Making a linear combination of Φ_+ and Φ_- we can get $\Phi_s=[\sin(kr)\cos(kt)]/r$. This field has no singularity - it is standing waves. Applying sphere integrals for the point r=0 we have: $\Phi_{in}(\Phi_s)=\Phi_{out}(\Phi_s)=k\cos(kt)$ which is equal $\Phi_s(t,r=0)$ as it was expected.

In general case, let us take a close look at the formulas (6') and (7'). We see that electrostatic potential when λ increases has 1/r dependence at infinity so it is zero at infinity. Electromagnetic waves potentials from (or to) the point source are $(1/r)e^{ik(r+/-t)}$. They also vanish at infinity but their derivatives multiplied on r do not decrease with radius. In each physical situation (the physical field Φ is given) we have four mathematical fields in the whole space (which can be obtained by the formulas (6),(6'),(7),(7') with big enough λ): fields of volume retarded and advanced potentials which satisfy to inhomogeneous wave equation and fields of sphere retarded and advanced potentials which satisfy to homogeneous wave equation. These mathematical fields can not be made into a physical reality separately except Φ_{in} (end also except the case when two of them equal to each other and other two are zero as in the case of charges moving without acceleration where retarded and advanced volume potentials coincide and sphere potentials are zero). If we have incoming physical radiation (take for example a plane wave) we can calculate $\Phi_{in}(t,x,y,z)$ and it coincides with incoming physical radiation. $\Phi_{out}(t,x,y,z)$ coincides with physical field at infinity but not around the currents.

Now, let us make another thought experiment. Suppose that in the above case with the charge resting at $z=-a$ we give at $t=0$ some initial data different initial data from $\Phi=1/\mathrm{sqrt}(x^2+y^2+(z+a)^2)$. Suppose we take $\Phi=0$. We can do this if we think that we can impose any data we like. Let us use the formula (4) to define the field at positive times. Our point of observation will be t, $x=y=z=0$. Since we want that sphere retarded potential to be taken at zero time, we assume that $t=\lambda$. On that condition Φ_{in} will be zero due to the initial data. If $t<a$ then $\Phi=0$ since the volume potential will be zero. If $t>a$ then $\Phi=1/a$ due to the volume potential. Taking $-t=\lambda$ and using formula (5) we get the same result for the negative times. Since a can be any positive value and due to the symmetry of the physical situation we can conclude that whole region of space outside of the light cone of the point where the charge is at $t=0$, $x=y=0$, $z=-a$ will be empty of field while the regions inside the forward and the backward light cones will be filled with the usual $1/a$ field. We can write down the field that we got:

$$\Phi=A^0=\theta(-u)/r \qquad (10)$$

where $u=r+t$ for $t<0$ and $u=r-t$ for $t>0$, and $\theta()$ is a step function. We got a scalar potential A^0 which depends of time and has jumps. The formulas (4) and (5) can fail due to discontinuity of A^0. But let us discuss this problem later and go further as if (4) and (5) work. Due to the time dependence of A^0 we should add some vector potential so that calibration:

$$A^k{}_{|k}=0 \qquad (11)$$

takes place:

$$\vec{A}=-t\vec{r}\theta(u)/r^3 \qquad (12)$$

We have:

$$\dot{A}^0+div(\vec{A})=\pm u\theta'(u)$$

since $\theta'(-u)=\theta'(u)$ is a delta function. $u\theta'(u)$ is zero in average. Let us be satisfied for now because we have not much of a choice. It can be checked that (10) and (12) satisfy to the wave equation. The only problem that could be here is that there was no given a source at $r=0$ for the vector potential (12). Let us calculate the electromagnetic field that corresponds to (10), (12):

$$\vec{E}=-\nabla A^0-\dot{\vec{A}}=\vec{r}/r^3; \quad \vec{H}=0$$

because $\theta(-u)+\theta(u)=1$. Surprisingly, the electromagnetic field does not change. Now, we are deep enough in trouble. Still, to show the technique, let us go further. In the case of positive time (point of observation is $t>0$, $x=y=z=0$) we can use the formula (5). The time of the sphere advanced potential will be $t+\lambda$. Suppose $t<a$ so that $\Phi=0$, but let us take λ big so that the volume potential will be $1/a$. From the formula (5) we can conclude that $\Phi_{out}=1/a$. That means that there exist some outgoing radiation which is connected to the point of observation. If we apply the formula (4) to the same point of observation with λ goes to infinity we will get that $\Phi_{in}=-1/a$. This incoming radiation cancels the field in space-like region around the hyperplane $t=0$ and then goes to infinity ($\Phi_{out}=\Phi_{in}$). In the coordinates where the charge resides in origin we have:

$$\Phi_{out}=(t,x,y,z)=\Phi_{in}=(t,x,y,z)=-\theta(u)/r \qquad (13)$$

It is rather easy to dump any radiation to infinity (Φ_{out}) but it is practically almost impossible to make a very specific (13) radiation coming from infinity to our experimental arrangement (Φ_{in}) to say nothing about the fact that this radiation exist only in electromagnetic potentials and does not reflected in electromagnetic field (we never even thought about such kind a possibility). This thought experiment can not be 100% reliable due to the disruptions of the involved functions. What I meant to show is that we can not prescribe initial data indiscriminately in the theory of electromagnetic field.

Though, there is the way out of the trouble. Let us instead of initial data prescribe incoming radiation $\Phi_{in}(t,x,y,z)$ in the whole space in the past and in the future. Considering very big so that all the sources are included in the integration in the volume potential we can use the formula (4) to find the field Φ. Knowing this field we can use the formula (5) to find outgoing radiation Φ_{out}. Let us rewrite the formula (5) in the order it should be used:

$$\Phi_{out}(t,x,y,z)=\Phi\ (t,x,y,z)-\Phi_{va}(t,x,y,z) \qquad (14)$$

The point of observation P has past and future light cones attached to it. Incoming radiation comes along the past light cone. It "picks up" retarded volume potential along its way to the point P. The outgoing radiation goes along the future light cone and also "picks up" advanced volume potential on its way to the future (t-inf.). This advanced volume potential contributes only to the outgoing radiation $\Phi_{out}(t,x,y,z)$. No way it can influence the field in the point of observation. Now we realize that in the theory of electromagnetic field we can not prescribe initial data at t=0. We can only prescribe the incoming radiation. Can we prescribe an arbitrary incoming radiation? We can just in principle but in reality we are restricted to what we can obtain by experimental devices. I believe that we can not create incoming radiation like (13). These devices can only create radiation by manipulating the currents assuming that there is no incoming radiation to these devices. So, we came to the conclusion that we always should start with the zero incoming radiation. After that we can consider some currents that create some outgoing radiation which, in turn, can be considered as incoming radiation for the another experimental arrangement.

The Problem of Irreversibility.

In the case where incoming radiation is zero the given set of currents can only emit radiation. Can we reverse the situation? We have: $\Phi=\Phi_{vr}+0$, $\Phi_{out}=\Phi-\Phi_{va}$ (we consider λ big enough). So, we have a mathematical field $\Phi_{out}(t,x,y,z)$ which coincides with the physical field far from the sources. Let us consider another physical field around the same sources with $\Phi'_{out}=0$: $\Phi'=\Phi_{va}+0$, $\Phi'_{in}=\Phi_{va}-\Phi_{vr}=-\Phi_{out}$, where Φ_{vr} , and Φ_{va} are the same because we use the same set of currents. We have a complete absorption here ($\Phi'_{out}=0$) but we have to provide a specific incoming radiation $\Phi'_{in}=-\Phi_{out}$ (remember that physical field now different - it coincides with volume advanced potentials). We reversed complete emission to complete absorption with the same set of currents. The problem being is how we produce this specific incoming radiation Φ'_{in} which looks like incoming to the center spherical wave? We have to have some coherent sources distributed over the sphere of a big radius. The laws of dynamics make this task very difficult. We can not expect incoming spherical waves in nature without our intelligent interference. Practically we can expect plane wave (or many incoherent plane waves) as incoming radiation only. This is the reason of irreversibility. With incoming plane wave we can expect that emission and absorption take place at the same time. We can imagine situation with incoming plane waves so strong that absorbed and emitted energy are equal to each other (for some definite set of currents).