

The Free Fall Law of Galileo is Only Approximately Correct

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Introduction

The free fall law of Galileo (FFLG for short) asserts that if two objects are left from the same point to fall freely in a gravitational field, then regardless of their masses both objects will cover the same distance during the same period of time. In particular, both objects will hit the surface of the attracting body at the same time. Employing Newton's laws of motion we show that the FFLG cannot stand as a law because of its limitations on predictive and conceptual grounds. The FFLG however is highly accurate when the masses of the falling objects are negligible in comparison with the gravitating object's mass.

The bases of this work can be found in most textbooks treating the two-body problem in classical mechanics [1,2]. Our rule here is confined to highlight the relevant aspects which reveal that the duration of free fall depends on the masses sum of the two gravitating objects. Equivalently, it is not true that: *two bodies with different masses falling freely in a gravitational field acquire the same acceleration at each point in space*. We shall also explore the great insensitivity of free fall time of light objects to considerable changes in their masses.

Two-body Central Force Problem.

Consider a closed system consisting of two particles of masses m_1 and m_2 interacting through a force that depend on the distance between them. Let S_0 be an inertial frame in which the positions of the particles are determined by \vec{r}_1 and \vec{r}_2 respectively. The position vector \vec{r} of the second particle relative to the first is given by

$$(1) \quad \vec{r} = \vec{r}_2 - \vec{r}_1.$$

It follows from (1) that the relative velocity and acceleration are

$$(2) \quad \vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}_2}{dt} - \frac{d\vec{r}_1}{dt} = \vec{v}_2 - \vec{v}_1,$$

$$(3) \quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}_2}{dt} - \frac{d\vec{v}_1}{dt} = \vec{a}_2 - \vec{a}_1.$$

By the Newton's law of motion the first (second) particle is subject to the force

$$\vec{f}_{12} = m_1 \vec{a}_1 \quad (\vec{f}_{21} = m_2 \vec{a}_2).$$

We impose on the force of interaction to obey the law of action and reaction;

that is $\vec{f}_{12} + \vec{f}_{21} = 0$, or

$$(4) \quad m_1 \vec{a}_1 + m_2 \vec{a}_2 = 0.$$

From (3) and (4) we have

$$(5) \quad \vec{a}_1 = -\frac{\mu}{m_1} \vec{a}, \quad \vec{a}_2 = \frac{\mu}{m_2} \vec{a},$$

with

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass of the system. From (5) we have

$$(6a) \quad -m_1 \vec{a}_1 = m_2 \vec{a}_2 = \mu \vec{a},$$

or

$$(6b) \quad -\vec{f}_{12} = \vec{f}_{21} = \mu\vec{a} = \vec{f}(r),$$

where

$$(7) \quad \vec{f}(r) = \vec{f}_{21}(r) = f(r)\vec{e}, \quad \text{with } \vec{e} = \vec{r}/r.$$

Conserved Physical Quantities

Integrating (4) yields

$$(8a) \quad m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_{10} + m_2\vec{v}_{20} \equiv \vec{P}_c,$$

or

$$(8b) \quad \vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{p}_{10} + \vec{p}_{20} = \vec{P}_c,$$

which expresses the conservation of the linear momentum of the closed system.

Solving (2) with (8a) yields

$$\vec{v}_1 = \frac{\vec{P}_c}{m_1 + m_2} - \frac{\mu}{m_1}\vec{v} \equiv \vec{V}_c - \frac{\mu}{m_1}\vec{v},$$

$$\vec{v}_2 = \frac{\vec{P}_c}{m_1 + m_2} + \frac{\mu}{m_2}\vec{v} \equiv \vec{V}_c + \frac{\mu}{m_2}\vec{v},$$

or

$$\vec{p}_1 = \mu\left(\frac{\vec{P}_c}{m_2} - \vec{v}\right), \quad \vec{p}_2 = \mu\left(\frac{\vec{P}_c}{m_1} + \vec{v}\right).$$

The **Kinetic energy** of the system is

$$T = \frac{1}{2} \frac{P_c^2}{m_1 + m_2} + \frac{1}{2} \mu v^2.$$

Integrating (8a) we obtain

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{P}_c t + m_1\vec{r}_{10} + m_2\vec{r}_{20} = \vec{P}_c t + (m_1 + m_2)\vec{R}_{c0}$$

where

$$m_1\vec{r}_{10} + m_2\vec{r}_{20} \equiv (m_1 + m_2)\vec{R}_{c0},$$

and \vec{R}_{c0} is the initial position of the center of mass. The **work** done by the internal forces on the particles when displaced by $d\vec{r}_1$ and $d\vec{r}_2$ respectively is

$$dW = \vec{f}_{12} \cdot d\vec{r}_1 + \vec{f}_{21} \cdot d\vec{r}_2 = \mu\vec{a} \cdot d\vec{r} = d\left(\frac{1}{2}\mu v^2\right) = dT$$

Potential energy: The force is derivable from the potential

$$V(r) = -\int \vec{f}(r) \cdot d\vec{r} = -\int f(r) dr,$$

and hence

$$dV = \nabla V \cdot d\vec{r} = -\vec{f}(r) \cdot d\vec{r} = -dT$$

It follows that the mechanical energy $E = T + V$ is conserved. The **angular momentum** about O

$$\vec{L} = \vec{r}_1 \times m_1\vec{v}_1 + \vec{r}_2 \times m_2\vec{v}_2 = \vec{R}_c \times \vec{P}_c + \mu\vec{r} \times \vec{v}$$

is conserved.

The above analysis is valid for any type of force that depends solely on the relative distance separating the two particles. For the gravitational force

$$(10) \quad \vec{f}(r) = -G \frac{m_1 m_2}{r^2} \frac{\vec{r}}{r},$$

we have from (6a)

$$\vec{a} = \frac{1}{\mu} \vec{f}(r) = -G \frac{m_1 + m_2}{r^2} \frac{\vec{r}}{r},$$

which shows, contrary to the FFLG, that the relative acceleration depends on the sum of the gravitating particles. Therefore, and for a given body (particle, planet, star, ...) with mass m_1 the relative acceleration depends surely on the mass m_2 of the other body through the sum $m_1 + m_2$.

Circular Orbits

We have from (6b)

$$(11) \quad m_1 \vec{a}_1 = G \frac{m_1 m_2 \vec{r}}{r^2} = -m_2 \vec{a}_2.$$

With the origin of the coordinates system is taken at the center of mass we have

$$(12) \quad \vec{r}_1 = -\frac{\mu}{m_1} \vec{r}, \quad \vec{r}_2 = -\frac{\mu}{m_2} \vec{r}.$$

The conditions for circular orbits are

$$(i) \quad \vec{r} \cdot \vec{v} = 0 \text{ at } t = t_0 \text{ (which equivalent to } \vec{r}_1 \cdot \vec{v}_1 = \vec{r}_2 \cdot \vec{v}_2 = 0)$$

$$(ii) \quad \frac{v_1^2}{r_1} = G \frac{m_2}{r^2}, \quad \frac{v_2^2}{r_2} = G \frac{m_1}{r^2}.$$

Thus for circular motion

$$(13) \quad \omega^2 r_1 = G \frac{m_2}{r^2}, \quad \omega^2 r_2 = G \frac{m_1}{r^2}.$$

Substituting (12) in (13) we obtain

$$(14) \quad \omega^2 r^3 = G(m_1 + m_2),$$

Or

$$(15) \quad \frac{r^3}{\tau^2} = \frac{G(m_1 + m_2)}{4\pi^2},$$

which is Kepler's third law. In this case the two particles rotate about the center of mass and each particle describes a circle (in the center of mass' frame) with the lighter particle describing the outer circle. Precisely, the particles m_1 and m_2 describe circles of radii given by (12) with the particles lying at each instant at opposite sides of the center of mass with constant separation r . Also, in a frame commoving with one particle and not rotating in S_0 the other particle executes a uniform circular motion with radius r .

Note that the latter results can be extended in a straightforward manner to the general case in which each particle describes an ellipse in the center of mass' frame. Indeed, and for the general bound motion in which

$$r = \frac{p}{1 + e \cos \theta}$$

with p is the ellipse parameter and $e < 1$ is the eccentricity, the particles m_1 and m_2 describe the ellipses

$$r_2 = \frac{\frac{\mu}{m_2} p}{1 + e \cos \theta}, \quad r_1 = \frac{\frac{\mu}{m_1} p}{1 + e \cos(\pi + \theta)}$$

that have the same eccentricity but different parameters:

$$p_1 = \frac{\mu}{m_1} p, \quad p_2 = \frac{\mu}{m_2} p.$$

The last facts might be applicable to determine the relative masses of two binary stars.

Special Cases:

For particles with equal masses: $m_1 = m_2 = m$, we have

$$\mu = \frac{1}{2}m, \quad r_1 = r_2 = \frac{1}{2}r,$$

$$-\vec{v}_1 = \vec{v}_2 = \frac{1}{2}\vec{v}, \quad -\vec{a}_1 = \vec{a}_2 = \frac{1}{2}\vec{a},$$

and the two particles trace the same circle in the center of mass frame opposing each other diametrically with the origin of their circular trajectory coinciding with the center of mass. In the general bound motion it is also true that the two particles with equal masses trace the same ellipse (in the center of mass frame) with their polar angles θ_1 and θ_2 maintaining a constant difference. i.e. $\theta_1 - \theta_2 \equiv \pi$.

For $m_1 = M \gg m_2 = m$ we have

$$\mu \approx m, \quad \vec{r}_1 = -\frac{\mu}{M}\vec{r} \approx 0, \quad \vec{r}_2 = \frac{\mu}{m}\vec{r} \approx \vec{r}$$

$$\vec{v}_1 \approx 0, \quad \vec{v}_2 \approx \vec{v}, \quad \vec{a}_1 \approx 0, \quad \vec{a}_2 \approx \vec{a}.$$

In this case the lighter mass orbits the heavier one, which remains approximately at rest at the center of mass, in a circular trajectory. Only in the latter case the FFLG holds accurate with an error that diminishes with the ratio m/M .

Head-On Fall (or Radial Fall):

This corresponds to the initial conditions:

$$\text{for } t = 0, \quad r = r_0 \quad \& \quad \vec{v} = 0 \quad (\text{or equivalently } \vec{v}_1 = \vec{v}_2 = 0).$$

In this case the particles fall towards each other with acceleration

$$(16) \quad \vec{a} = \frac{1}{\mu}\vec{f}(r) = -G \frac{m_1 + m_2}{r^2} \frac{\vec{r}}{r},$$

which depends also on the mass of the falling particle (which could be chosen either one). To integrate equation (16) we multiply both sides of the equation:

$$(17) \quad \frac{d^2r}{dt^2} = -G \frac{m_1 + m_2}{r^2}$$

by $2dr/dt$ and integrate to obtain

$$(18) \quad \left(\frac{dr}{dt}\right)^2 = \frac{2G(m_1 + m_2)}{r} + C.$$

By the initial conditions,

$$(19) \quad C = -\frac{2G(m_1 + m_2)}{r_0},$$

and hence

$$(20) \quad \left(\frac{dr}{dt}\right)^2 = 2G(m_1 + m_2) \left(\frac{1}{r} - \frac{1}{r_0}\right),$$

Or

$$(21) \quad \frac{dr}{dt} = -\sqrt{\frac{2G(m_1 + m_2)}{r_0}} \sqrt{\frac{r_0 - r}{r}}.$$

The time dependence of the relative distance is given by

$$(22) \quad \int \sqrt{\frac{r}{r_0 - r}} dr = - \int \sqrt{\frac{2G(m_1 + m_2)}{r_0}} dt.$$

The integral on the left hand-side is implemented through setting $r_0 - r = z^2$ and then $z = \sqrt{r_0} \sin t$. The result of integration is

$$(23) \quad \sqrt{r(r_0 - r)} + r_0 \sin^{-1} \sqrt{\frac{r_0 - r}{r_0}} = \sqrt{\frac{2G(m_1 + m_2)}{r_0}} t + 0.$$

The duration of the free fall

$$(24) \quad t = \sqrt{\frac{r_0}{2G(m_1 + m_2)}} \left\{ \sqrt{r(r_0 - r)} + r_0 \sin^{-1} \sqrt{\frac{r_0 - r}{r_0}} \right\}$$

depends on the mass of the falling particle, say m_2 , through its sum with the gravitating mass m_1 .

When the free fall covers only a small distance $h = r_0 - r \ll r$ then the following approximations hold

$$\begin{aligned} r_0 &\approx r, & \sqrt{r_0} \sqrt{r(r_0 - r)} &\approx r \sqrt{h}, \\ \sqrt{r_0} r_0 \sin^{-1} \sqrt{\frac{r_0 - r}{r_0}} &\approx \sqrt{r_0} r_0 \cdot \sqrt{\frac{r_0 - r}{r_0}} = r_0 \sqrt{r_0 - r} \approx r \sqrt{h}. \end{aligned}$$

Substituting in (24) we obtain

$$t \approx \frac{2r \sqrt{h}}{\sqrt{2G(m_1 + m_2)}}$$

from which we find the familiar form which gives the relation between the distance travelled by a particle in a constant uniform acceleration field as a function of time:

$$h = \frac{1}{2} \frac{G(m_1 + m_2)}{r^2} t^2 = \frac{1}{2} a t^2,$$

In the last equation we used (18) to set $G(m_1 + m_2)/r^2 = a$, which is approximately constant over a small change in r when r is much greater than h . Note that r_0 and r represent the initial and final distances between the centers of the two bodies. If the radius of the second body is negligible in comparison with the radius r of the first and g is the acceleration at the surface of the first body, then we may set $a = g$ in the latter equation, and the situation encountered is identical to the free fall near the earth's surface.

Degree of Accuracy of the Free Fall Law

The free fall's duration given by (24) can be written as follows:

$$(25) \quad t = \frac{1}{\sqrt{1 + m_2/m_1}} \sqrt{\frac{r_0}{2Gm_1}} \left\{ \sqrt{r(r_0 - r)} + r_0 \sin^{-1} \sqrt{\frac{r_0 - r}{r_0}} \right\} \\ = \frac{T}{\sqrt{1 + m_2/m_1}},$$

where

$$(26) \quad T = \sqrt{\frac{r_0}{2Gm_1}} \left\{ \sqrt{r(r_0 - r)} + r_0 \sin^{-1} \sqrt{\frac{r_0 - r}{r_0}} \right\}$$

is the free fall duration of a minute object with a negligible mass (call it nothingness). In fact, it is the fall time of nothingness T what is adopted in the GL of FF; it results from t in (24) after setting $m_2 = 0$.

Equation (25) reduces to

$$(27) \quad t = \frac{T}{\sqrt{2}} \quad \text{for } m_1 = m_2,$$

and to

$$(28) \quad t \approx \left(1 - \frac{m_2}{2m_1}\right) T \quad \text{for } m_1 \gg m_2.$$

Equation (28) shows that the time of fall t of a mass m_2 , with $m_2/m_1 \ll 1$, is less by $(m_2/2m_1)T$ from nothingness fall time. This difference can be extremely trivial if m_1 is sufficiently large in comparison with m_2 .

Example: Free Fall Towards the Earth.

We consider here the free fall corresponding to the gravitational force between the earth and objects that have much smaller mass. We neglect of course the effect of air resistance. With the earth's mass: $m_1 \approx 6 \times 10^{24} kg$, the masses of a neutron or a feather can be counted nothingness. Corresponding to a fall duration of nothingness, $T = 100$ seconds, the fall time of a mass m_2 falling freely towards the earth from the same point at which the nothingness starts will be

$$t = \left(1 - \frac{m_2}{2m_1}\right) T = \left(1 - \frac{m_2 kg}{12 \times 10^{24} kg}\right) \times 100 sec.$$

For a difference $(m_2/2m_1)T$ not exceeding 10^{-19} seconds = 10^{-10} ns we have to choose the falling mass such that

$$\frac{m_2}{2m_1} T = \frac{m_2}{12 \times 10^{24}} \times 100 sec \leq 10^{-19} seconds$$

which yields

$$m_2 \leq 12 \times 10^3 kg = 12 \text{ Ton}.$$

Thus any falling body with mass not exceeding 12 Ton will lack behind the nothingness in a fall trip of duration 100 seconds by less than 10^{-10} ns. The delay period for a body of mass 120 kg will be 10^{-12} ns, and if the fall time is only 5 seconds then the delay period will be about 4×10^{-14} ns.

On the other hand, and if Earth and Venus, that have approximately equal masses, fall freely towards each other (in the absence of the sun and other planets) then there corresponds to $T = 100$ seconds the fall period

$$t = \frac{T}{\sqrt{2}} = \frac{100 seconds}{\sqrt{2}} \approx 70 \text{ seconds}.$$

Conclusion

In FFLG a qualitative distinction is drawn between the falling object and the gravitating one. The falling object is completely passive in the sense that it has no effect on the gravitating object, which alone sets up a gravitational field that assigns to every point in space an acceleration depending only on the radial distance of that point from the gravitating body and on the mass of the latter. The

asymmetry between the gravitating mass and the falling one is clearly artificial and illogical. This becomes apparent when the falling mass is of the magnitude of the gravitating one. The FFLG is highly accurate for an almost closed system of masses out of which only one mass, say m_1 , is much larger than the rest. In this case FFLG describes accurately the fall of all other masses in the gravitational field of m_1 .

Gratitude

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References

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