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Black body radiation in quantum Statistical Mechanics distribution

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Introduction

All surfaces at a finite temperature emit electromagnetic radiation, but at room temperature this energy emission is weak and almost entirely distributed in the far infrared spectrum. With increasing temperature the total radiated power increases rapidly, proportional to the fourth power of the absolute temperature (Stefan's Law).

Black body radiation (An ideal gas of photons):

The modern theory of black body radiation, which was developed by Max Plank in 1900, gave birth to the subject of quantum physics. Plank's analysis (1900-1901) changed the course of science, what, then is a black body and How does it radiate?

A black body is defined as one which absorbs all the radiation which is incident upon it none is reflected. when it is cold it looks black any radiation which incident no the body is absorbed the same body when heated will glow if you were to look through a tiny hole in a furnace then the reddish glow you would see be fair approximation to the visible portion of black body radiation the visible part of the spectrum is wavelengths ranging from 400nm to 750 nm radiation of wavelength outside this range cannot be seen by the naked eye what you see is only a small part of total spectrum of radiation .

Derivation of the Planck distribution:

From one point of view, we can analyses the electromagnetic field in a box or cavity in terms of a lot of harmonic oscillators, treating each mode of oscillation according to quantum mechanics as a harmonic oscillator. From a different point of view, we can analyses the same physics in terms of identical Bose particles. And the results of both ways of working are always in exact agreement. there is no way to make up your mind whether the electromagnetic field is really to be described as a quantized harmonic oscillator or giving how many photons are in each condition . The two view turn out to be mathematically identical

Richard Feynman (1965)

The classical wave equation is

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

We shall apply statistical thermodynamic and electromagnetic radiation enclosed in fixed volume V and the temperature T the experimental system is obtain by making a cavity in any material the equilibrium we will have cavity filled with electromagnetic radiation the wave electromagnetic we can find the momentum and energy that are function of spin the measles particles are call 0photons the Maxwell's function can be write is

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$$\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Consider a harmonic electromagnetic wave of unite amplitude traveling with velocity c in the positive direct x . Mathematically this is described by

$$E(x,t)=\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)$$

And is call traveling wave and λ is wavelength of wave

$$E(x,t)=\sin\left((kx-\omega t)\right)$$

We study the wave function in the appropriate boundary condition is that

$$\phi(x,t)=\sin\left((kx-\omega t)\right)+\sin\left((kx+\omega t)\right)=2\sin(kx)\cos(\omega t)$$

By the boundary condition give

$$k_x=\frac{n\pi}{L}$$

If we let components of the wave vector k_x, k_y, k_z

$$k_x=\frac{\pi}{L}n_x \qquad k_y=\frac{\pi}{L}n_y \qquad k_z=\frac{\pi}{L}n_z$$

We are going to need an expression for the number of standing wave with energy between ε and $\varepsilon+d\varepsilon$

$$k^2=k_x^2+k_y^2+k_z^2=\frac{\pi^2}{L^2}(n_x^2+n_y^2+n_z^2)$$

The equation would still be that of a simple harmonic oscillator. the following analysis applies to both standing and traveling waves moreover, the simple harmonic oscillator equation is obtained for other waves, such as waves on the surface of a liquid in which the angular frequency $\omega(k)$, doses not very linearly with k in generally $\omega(k)$ is arbitrary function of the waves vector K , but we shall simplify matters by assuming that it only depends on the magnitude of the wave vector , that is k

What are the quantum mechanics energy eigenvalue for a harmonic oscillator? if we treat the wave as if they were simple harmonics oscillators the energy eigenvalue

$$E_n=\hbar\omega(n+\frac{1}{2})$$

The separation between neighboring energy levels is $\hbar\omega$ so the energy is quantized (remember that $\hbar=h/2\pi$ so that $h\nu=\hbar\omega$ the number n represents the quantum number of the harmonic oscillator in this picture it is not associated with the number of particles

<u>waves as particles :</u>

Planck quantized the oscillators which he imagined to exist in the wall of the cavity. He was unaware that his quantization proposal could be applies to oscillation of the classical radiation field itself this was the point of view which was put forward by Einstein (1905) in the paper in which he proposed an explained for photoelectric effect As Einstein put it ' the energy of a light ray spreading from a light source is not continuously distributed over an increasing space but consists of a finite number of energy quanta which move without diving and which can only be produced and absorbed as complete unite He pictured the radiation as made up of particles with each particles having an energy when it is in a standing wave state if

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there are no particle the energy in the stat is zero one particle $h\nu$ and two particle is $2h\nu$ to n particle

Einstein was awarded the Nobel Prize , not for his theory of special relativity but for suggestion that light could be considered as if it were made of particles a truly revolutionary suggestion bearing in mind the success of Maxwell's theory of electromagnetism

During the slow acceptance of Einstein's quantum hypothesis the question arose whether other waves have quantized energy levels .

The Raleigh – Jeans theory

The black body consists of electromagnetic radiation in thermal equilibrium with the walls of cavity. When they are in thermal equilibrium , the average rate of emission of radiation by the walls equals their average rate of absorption of radiation The condition for thermal equilibrium is that the temperature of the walls is equal to the temperature of radiation .But what do we mean by ideal that radiation has a temperature ?

One can imagine that the material of the walls contains charged particles which oscillated about their equilibrium positions .(it does not matter what the walls are made of , so we can mentally imagine them to consist of oscillating charge) As was know from Maxwell's work on electromagnetism a moving charge radiates an electromagnetic wave . We imagine the walls to be full of charged particles jiggling about coupled to electromagnetic standing.

Suppose each standing wave mode of the electromagnetic field is coupled to an oscillator in the wall which oscillates with the same frequency as the standing wave each oscillator has two degrees of freedom one for kinetic energy and the other for potential energy and so it has an average energy of kT according to the equipartition theory in thermal equilibrium the average energy of the oscillator and the average energy of standing wave mode of the electromagnetic field

Thus the number of standing wave with magnitude of vector less than k

$$\phi(k) = \frac{\pi}{6}\left(\frac{Lk}{\pi}\right)^3 = \frac{Vk^3}{6\pi^2}$$

So

$$\omega(k) = \frac{d\phi}{dk}dk = \frac{Vk^2}{2\pi^2}dk$$

By the relation give

$$\varepsilon = \hbar ck , k = \frac{\varepsilon}{\hbar c} , dk = \frac{1}{\hbar c}d\varepsilon$$

Give

$$\omega(\varepsilon)d\varepsilon = \frac{V\varepsilon^2}{2\pi^2\hbar^3c^3}d\varepsilon$$

There are to polarization with energy give

$$\omega(\varepsilon)d\varepsilon = \frac{V\varepsilon^2}{\pi^2\hbar^3c^3}d\varepsilon$$

The last result by integral given infinity

$$\int_0^\infty \frac{V\varepsilon^2}{2\pi^2\hbar^3c^3}d\varepsilon = \infty$$

Planck distribution:

Now we obtain the Planck's distribution by the total energy system is give

$$E(\{n_k\}) = \sum_k \varepsilon_k n_k$$

By Boltzamann distribution the partition function give

$$Q(V,T) = \sum_{n_k} e^{-\beta \sum_k \varepsilon_k n_k} = \sum_{n_k} (e^{-\beta \varepsilon_1 n_k})(e^{-\beta \varepsilon_2 n_k}).......$$

$$Q(V,T) = \prod_k \sum_{n_k} (e^{-\beta \varepsilon_k})^n$$

By this relation

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1-z}$$

The equation can be write

$$Q(V,T) = \prod_k \sum_{n_k} (e^{-\beta \varepsilon_k})^n = \prod_k (1-e^{-\beta \varepsilon_k})^{-1}$$

Take ln to side

$$\ln Q = -\sum_k \ln(1-e^{-\beta \varepsilon_k})$$

By the introduction the density states and treating E to be continuous variable

$$\ln Q = -\frac{V}{\pi^2 \hbar^3 c^3} \int_0^{\infty} \varepsilon^2 \ln(1-e^{-\beta \varepsilon_k}) d\varepsilon$$

We can write the integral by

$$\ln(1-e^{-\beta \varepsilon_k}) = \sum_n \frac{e^{-\beta n \varepsilon_k}}{n}$$

But

$$\varepsilon = \frac{y}{\beta n}, d\varepsilon = \frac{dy}{\beta n}$$

The integral

$$\ln Q = -\frac{V}{\pi^2 \hbar^3 c^3} \int_0^{\infty} \varepsilon^2 \sum_n \frac{e^{-\beta n \varepsilon_k}}{n} d\varepsilon$$

The last equation give lnQ like

$$\ln Q = -\frac{2V}{\pi^2 (\hbar \beta c)^3} \sum_n \frac{1}{n^4}$$

By definition of zeta function $\zeta(4) = \pi^4/90$

$$\ln Q = -\frac{2V}{\pi^2 (\hbar \beta c)^3} \frac{\pi^4}{90} = \frac{\pi^2 V k^3 T^3}{45 c^3 \hbar^2}$$

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By the relation in chapter 1

$$\bar{E} = -\frac{\partial \ln Q}{\partial \beta} = kT^2 \frac{\partial \ln Q}{\partial T}$$

In the finally we get E by derivative lnQ

$$\bar{E} = \frac{\pi^2 V k^4}{15 c^3 \hbar^2} T^4$$

This result can be used to derive the Stefan – Boltzmann law by analogy then cE/4V is the energy incident per unit area per unit time on the wall of the enclosure containing the radiation .thus if one cuts a small hole of unite area in the wall , the energy radiated per unite time

$$R = \frac{\pi^2 k^4}{60 c^2 \hbar^2} T^4 = \alpha T^4$$

Where α **is know as the Stefan- Boltzmann constant**
Now we can fin the mean of energy by partition function give

$$\bar{E} = -\frac{\partial \ln Q}{\partial \beta} = \frac{\varepsilon_k e^{-\beta \varepsilon_k}}{1 - e^{-\beta \varepsilon_k}} = \frac{h\nu}{e^{h\nu / kT} - 1}$$

We can find the I by

$$I_\nu = \frac{h}{\pi^2 c^3} \frac{\nu^3}{e^{h\nu / kT} - 1}$$

By the relation $\nu = c/\lambda$ **and** $d\nu = -(c/\lambda^2) d\lambda$ **we get now**

$$I_\lambda = \frac{hc}{\pi^2 \lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

The maximize frequency give $dI_\lambda / d\lambda_{\max} = 0$ **we get**

$$\lambda_{\max} T = 0.0029 mK$$

According to statistical mechanics the probability distribution over the energy levels of a particular mode is given by:

$$P_r = \frac{\exp\left(-\beta E\left(r\right)\right)}{Z\left(\beta\right)}.$$

Here

$$\beta \overset{\text{def}}{=} 1/\left(kT\right).$$

The denominator *Z*(β), is the partition function of a single mode and makes P_r properly normalized:

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$$Z(\beta) = \sum_{r=0}^{\infty} \exp\left[-\beta E(r)\right] = \frac{1}{1-\exp\left[-\beta\varepsilon\right]}.$$

Here we have implicitly defined

$$\varepsilon \overset{\text{def}}{=} \frac{hc}{2L}\sqrt{n_1^2+n_2^2+n_3^2},$$

which is the energy of a single photon, the average energy in a mode can be expressed in terms of the partition function:

$$\langle E\rangle = -\frac{d\log(Z)}{d\beta} = \frac{\varepsilon}{\exp(\beta\varepsilon)-1}.$$

This formula is a special case of the general formula for particles obeying Bose-Einstein statistics. Since there is no restriction on the total number of photons, the chemical potential is zero.

The total energy in the box now follows by summing $\langle E\rangle$ over all allowed single photon states. This can be done exactly in the thermodynamic limit as *L* approaches infinity. In this limit, ε becomes continuous and we can then integrate $\langle E\rangle$ over this parameter. To calculate the energy in the box in this way, we need to evaluate how many photon states there are in a given energy range. If we write the total number of single photon states with energies between ε and $\varepsilon + d\varepsilon$ as $g(\varepsilon)\,d\varepsilon$, where $g(\varepsilon)$ is the density of states which we'll evaluate in a moment, then we can write:

$$U = \int_0^{\infty} \frac{\varepsilon}{\exp(\beta\varepsilon)-1} g(\varepsilon)\,d\varepsilon. \qquad (2)$$

To calculate the density of states we rewrite equation (1) as follows:

$$\varepsilon \overset{\text{def}}{=} \frac{hc}{2L} n,$$

where *n* is the norm of the vector $\vec{n} = (n_1, n_2, n_3)$:

$$n = \sqrt{n_1^2+n_2^2+n_3^2}.$$

For every vector *n* with integer components larger than or equal to zero there are two photon states. This means that the number of photon states in a certain region of n-space is twice the volume of that region. An energy range of $d\varepsilon$ corresponds to shell of thickness $dn = (2L/hc)d\varepsilon$ in n-space. Because the components of $\vec{n}$ have to be positive, this shell spans an octant of a sphere. The number of photon states $g(\varepsilon)\,d\varepsilon$ in an energy range $d\varepsilon$ is thus given by:

$$g(\varepsilon)\,d\varepsilon = 2\frac{1}{8}4\pi n^2\,dn = \frac{8\pi L^3}{h^3c^3}\varepsilon^2\,d\varepsilon.$$

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Inserting this in Eq. (2) gives:

$$U = L^3 \frac{8\pi}{h^3c^3} \int_0^\infty \frac{\varepsilon^3}{\exp(\beta\varepsilon) - 1} \, d\varepsilon. \qquad (3)$$

From this equation one easily derives the spectral energy density as a function of frequency *u*(ν,*T*) and as a function of wavelength *u*(λ,*T*):

$$\frac{U}{L^3} = \int_0^\infty u(\nu, T) \, d\nu,$$

where:

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}.$$

***u*(ν,*T*) is known as the black body spectrum. It is a spectral energy density function with units of energy per unit frequency per unit volume.**

And:

$$\frac{U}{L^3} = \int_0^\infty u(\lambda, T) \, d\lambda,$$

where

$$u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}.$$

This is also a spectral energy density function with units of energy per unit wavelength per unit volume. Integrals of this type for Bose and Fermi gases can be expressed in terms of polylogarithms. In this case, however, it is possible to calculate the integral in closed form using only elementary functions. Substituting

$$\varepsilon = kTx,$$

in Eq. (3), makes the integration variable dimensionless giving:

$$u(T) = \frac{8\pi (kT)^4}{(hc)^3} J,$$

where *J* is given by:

$$J = \int_0^\infty \frac{x^3}{\exp(x) - 1} \, dx = \frac{\pi^4}{15}.$$

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We prove this result in the Appendix below. The total electromagnetic energy inside the box is thus given by:

$$\frac{U}{V} = \frac{8\pi^5 (kT)^4}{15(hc)^3},$$

where $V = L^3$ is the volume of the box. (Note - This is not the Stefan-Boltzmann law which is the total energy radiated by a black body. See that article for an explanation.)

Since the radiation is the same in all directions, and propagates at the speed of light (c), the spectral radiance (energy/time/area/solid angle/frequency) of radiation exiting the small hole is

$$I(\nu,T) = \frac{u(\nu,T)\,c}{4\pi},$$

which yields

$$I(\nu,T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}.$$

It can be converted to an expression for $I'(\lambda,T)$ in wavelength units by substituting ν by c / λ and evaluating

Electromagnetic Waves in a Cubical Cavity

Electromagnetic standing waves in a cavity at equilibrium with its surroundings cannot take just any path. They must satisfy the wave equation in three dimensions:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

The solution to the wave equation must give zero amplitude at the walls, since a non-zero value would dissipate energy and violate our supposition of equilibrium. To form a standing wave, the reflection path around the cavity must produce a closed path. The boundary conditions can be met with a solution of the form:

$$E = E_0 \sin\frac{n_1 \pi x}{L} \sin\frac{n_2 \pi y}{L} \sin\frac{n_3 \pi z}{L} \sin\frac{2\pi c t}{\lambda}$$

Substituting this solution into the wave equation above gives

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$$\left[\frac{n_1\pi}{L}\right]^2+\left[\frac{n_2\pi}{L}\right]^2+\left[\frac{n_3\pi}{L}\right]^2=\left[\frac{2\pi}{\lambda}\right]^2$$

which simplifies to

$$n_1^2+n_2^2+n_3^2=\frac{4L^2}{\lambda^2}$$

Cavity Modes

A mode for an electromagnetic wave in a cavity must satisfy the condition of zero electric field at the wall. If the mode is of shorter wavelength, there are more ways you can fit it into the cavity to meet that condition. Careful analysis by Rayleigh and Jeans showed that the number of modes was proportional to the frequency squared.

Planck Radiation Formula

From the assumption that the electromagnetic modes in a cavity were quantized in energy with the quantum energy equal to Planck's constant times the frequency, Planck derived a radiation formula. The average energy per "mode" or "quantum" is the energy of the quantum times the probability that it will be occupied (the Einstein-Bose distribution function):

$$\langle E\rangle=\frac{h\upsilon}{e^{h\upsilon/kT}-1}$$

This average energy times the density of such states, expressed in terms of either frequency or wavelength

$$\rho(\nu)=\frac{dn_s}{d\nu}=\frac{8\pi}{c^3}\nu^2 \qquad \rho(\lambda)=\frac{dn_s}{d\lambda}=\frac{8\pi}{\lambda^4}$$

gives the energy density , the Planck radiation formula

The Planck radiation formula is an example of the distribution of energy according to Bose-Einstein statistics. The above expressions are obtained by multiplying the density of states in terms of frequency or wavelength times the photon energy times the Bose-Einstein distribution function with normalization constant A=1.

To find the radiated power per unit area from a surface at this temperature, multiply the energy density by c/4. The density above is for thermal equilibrium, so setting inward=outward gives a factor of 1/2 for the radiated power outward. Then one must average over all angles, which gives another factor of 1/2 for the angular dependence which is the square of the cosine.

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