

# The Hardy-Littlewood prime $k$ -tuple conjecture is false

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## Abstract

Using Jiang function we prove Jiang prime  $k$ -tuple theorem. We prove that the Hardy-Littlewood prime  $k$ -tuple conjecture is false. Jiang prime  $k$ -tuple theorem can replace the Hardy-Littlewood prime  $k$ -tuple conjecture.

### (A) Jiang prime $k$ -tuple theorem [1, 2].

We define the prime  $k$ -tuple equation

$$p, p + n_i, \quad (1)$$

where  $2 \mid n_i, i = 1, \dots, k-1$ .

we have Jiang function [1, 2]

$$J_2(\omega) = \prod_P (P-1 - \chi(P)), \quad (2)$$

where  $\omega = \prod_P P$ ,  $\chi(P)$  is the number of solutions of congruence

$$\prod_{i=1}^{k-1} (q + n_i) \equiv 0 \pmod{P}, \quad q = 1, \dots, P-1. \quad (3)$$

If  $\chi(P) < P-1$  then  $J_2(\omega) \neq 0$ . There exist infinitely many primes  $P$  such that each of  $P + n_i$  is prime. If  $\chi(P) = P-1$  then  $J_2(\omega) = 0$ . There exist finitely many primes  $P$  such that each of  $P + n_i$  is prime.  $J_2(\omega)$  is a subset of Euler function  $\phi(\omega)$  [2].

If  $J_2(\omega) \neq 0$ , then we have the best asymptotic formula of the number of prime  $P$  [1, 2]

$$\pi_k(N, 2) = \left| \left\{ P \leq N : P + n_i = \text{prime} \right\} \right| \sim \frac{J_2(\omega) \omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N} = C(k) \frac{N}{\log^k N} \quad (4)$$

$$\phi(\omega) = \prod_P (P-1),$$

$$C(k) = \prod_P \left( 1 - \frac{1 + \chi(P)}{P} \right) \left( 1 - \frac{1}{P} \right)^{-k} \quad (5)$$

**Example 1.** Let  $k = 2, P, P + 2$ , twin primes theorem.

From (3) we have

$$\chi(2) = 0, \quad \chi(P) = 1 \quad \text{if } P > 2, \quad (6)$$

Substituting (6) into (2) we have

$$J_2(\omega) = \prod_{P \geq 3} (P - 2) \neq 0 \quad (7)$$

There exist infinitely many primes  $P$  such that  $P + 2$  is prime. Substituting (7) into (4) we have the best asymptotic formula

$$\pi_k(N, 2) = \left| \{P \leq N : P + 2 = \text{prime}\} \right| \sim 2 \prod_{P \geq 3} \left(1 - \frac{1}{(P-1)^2}\right) \frac{N}{\log^2 N}. \quad (8)$$

**Example 2.** Let  $k = 3, P, P + 2, P + 4$ .

From (3) we have

$$\chi(2) = 0, \quad \chi(3) = 2 \quad (9)$$

From (2) we have

$$J_2(\omega) = 0. \quad (10)$$

It has only a solution  $P = 3, P + 2 = 5, P + 4 = 7$ . One of  $P, P + 2, P + 4$  is always divisible by 3.

**Example 3.** Let  $k = 4, P, P + n$ , where  $n = 2, 6, 8$ .

From (3) we have

$$\chi(2) = 0, \chi(3) = 1, \chi(P) = 3 \quad \text{if } P > 3. \quad (11)$$

Substituting (11) into (2) we have

$$J_2(\omega) = \prod_{P \geq 5} (P - 4) \neq 0, \quad (12)$$

There exist infinitely many primes  $P$  such that each of  $P + n$  is prime.

Substituting (12) into (4) we have the best asymptotic formula

$$\pi_4(N, 2) = \left| \{P \leq N : P + n = \text{prime}\} \right| \sim \frac{27}{3} \prod_{P \geq 5} \frac{P^3(P-4)}{(P-1)^4} \frac{N}{\log^4 N} \quad (13)$$

**Example 4.** Let  $k = 5, P, P + n$ , where  $n = 2, 6, 8, 12$ .

From (3) we have

$$\chi(2) = 0, \chi(3) = 1, \chi(5) = 3, \chi(P) = 4 \quad \text{if } P > 5 \quad (14)$$

Substituting (14) into (2) we have

$$J_2(\omega) = \prod_{P \geq 7} (P-5) \neq 0 \quad (15)$$

There exist infinitely many primes  $P$  such that each of  $P+n$  is prime. Substituting (15) into (4) we have the best asymptotic formula

$$\pi_5(N, 2) = \left| \{P \leq N : P+n = \text{prime}\} \right| \sim \frac{15^4}{2^{11}} \prod_{P \geq 7} \frac{(P-5)P^4}{(P-1)^5} \frac{N}{\log^5 N} \quad (16)$$

**Example 5.** Let  $k = 6$ ,  $P$ ,  $P+n$ , where  $n = 2, 6, 8, 12, 14$ .

From (3) and (2) we have

$$\chi(2) = 0, \chi(3) = 1, \chi(5) = 4, J_2(5) = 0 \quad (17)$$

It has only a solution  $P = 5$ ,  $P+2 = 7$ ,  $P+6 = 11$ ,  $P+8 = 13$ ,  $P+12 = 17$ ,  $P+14 = 19$ . One of  $P+n$  is always divisible by 5.

**(B) The Hardy-Littlewood prime  $k$ -tuple conjecture[3-14].**

**This conjecture is generally believed to be true, but has not been proved (Odlyzko et al. 1999).**

We define the prime  $k$ -tuple equation

$$P, P+n_i \quad (18)$$

where  $2|n_i, i = 1, \dots, k-1$ .

In 1923 Hardy and Littlewood conjectured the asymptotic formula

$$\pi_k(N, 2) = \left| \{P \leq N : P+n_i = \text{prime}\} \right| \sim H(k) \frac{N}{\log^k N}, \quad (19)$$

where

$$H(k) = \prod_P \left( 1 - \frac{\nu(P)}{P} \right) \left( 1 - \frac{1}{P} \right)^{-k} \quad (20)$$

$\nu(P)$  is the number of solutions of congruence

$$\prod_{i=1}^{k-1} (q+n_i) \equiv 0 \pmod{P}, \quad q = 1, \dots, P. \quad (21)$$

From (21) we have  $\nu(P) < P$  and  $H(k) \neq 0$ . For any prime  $k$ -tuple equation there exist infinitely many primes  $P$  such that each of  $P+n_i$  is prime, which is false.

**Conjecture 1.** Let  $k = 2, P, P+2$ , twin primes theorem

From (21) we have

$$\nu(P) = 1 \quad (22)$$

Substituting (22) into (20) we have

$$H(2) = \prod_P \frac{P}{P-1} \quad (23)$$

Substituting (23) into (19) we have the asymptotic formula

$$\pi_2(N, 2) = \left| \{P \leq N : P+2 = \text{prime}\} \right| \sim \prod_P \frac{P}{P-1} \frac{N}{\log^2 N} \quad (24)$$

which is false see example 1.

**Conjecture 2.** Let  $k = 3, P, P+2, P+4$ .

From (21) we have

$$\nu(2) = 1, \nu(P) = 2 \text{ if } P > 2 \quad (25)$$

Substituting (25) into (20) we have

$$H(3) = 4 \prod_{P \geq 3} \frac{P^2(P-2)}{(P-1)^3} \quad (26)$$

Substituting (26) into (19) we have asymptotic formula

$$\pi_3(N, 2) = \left| \{P \leq N : P+2 = \text{prime}, P+4 = \text{prim}\} \right| \sim 4 \prod_{P \geq 3} \frac{P^2(P-2)}{(P-1)^3} \frac{N}{\log^3 N} \quad (27)$$

which is false see example 2.

**Conjecture 3.** Let  $k = 4, P, P+n$ , where  $n = 2, 6, 8$ .

From (21) we have

$$\nu(2) = 1, \nu(3) = 2, \nu(P) = 3 \text{ if } P > 3 \quad (28)$$

Substituting (28) into (20) we have

$$H(4) = \frac{27}{2} \prod_{P > 3} \frac{P^3(P-3)}{(P-1)^4} \quad (29)$$

Substituting (29) into (19) we have asymptotic formula

$$\pi_4(N, 2) = \left| \{P \leq N : P+n = \text{prime}\} \right| \sim \frac{27}{2} \prod_{P > 3} \frac{P^3(P-3)}{(P-1)^4} \frac{N}{\log^4 N} \quad (30)$$

Which is false see example 3.

**Conjecture 4.** Let  $k = 5, P, P+n$ , where  $n = 2, 6, 8, 12$

From (21) we have

$$\nu(2) = 1, \nu(3) = 2, \nu(5) = 3, \nu(P) = 4 \text{ if } P > 5 \quad (31)$$

Substituting (31) into (20) we have

$$H(5) = \frac{15^4}{4^5} \prod_{P>5} \frac{P^4(P-4)}{(P-1)^5} \quad (32)$$

Substituting (32) into (19) we have asymptotic formula

$$\pi_5(N, 2) = \left| \{P \leq N : P+n = \text{prime}\} \right| \sim \frac{15^4}{4^5} \prod_{P>5} \frac{P^4(P-4)}{(P-1)^5} \frac{N}{\log^5 N} \quad (33)$$

Which is false see example 4.

**Conjecture 5.** Let  $k = 6$ ,  $P$ ,  $P+n$ , where  $n = 2, 6, 8, 12, 14$ .

From (21) we have

$$\nu(2) = 1, \nu(3) = 2, \nu(5) = 4, \nu(P) = 5 \text{ if } P > 5 \quad (34)$$

Substituting (34) into (20) we have

$$H(6) = \frac{15^5}{2^{13}} \prod_{P>5} \frac{(P-5)P^5}{(P-1)^6} \quad (35)$$

Substituting (35) into (19) we have asymptotic formula

$$\pi_6(N, 2) = \left| \{P \leq N : P+n = \text{prime}\} \right| \sim \frac{15^5}{2^{13}} \prod_{P>5} \frac{(P-5)P^5}{(P-1)^6} \frac{N}{\log^6 N} \quad (36)$$

which is false see example 5.

**Conclusion.** The Hardy-Littlewood prime  $k$ -tuple conjecture is false. The tool of additive prime number theory is basically the Hardy-Littlewood prime tuples conjecture. Jiang prime  $k$ -tuple theorem can replace Hardy-Littlewood prime  $k$ -tuple Conjecture. There cannot be really modern prime theory without Jiang function.

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