The Simplest Proofs of Both Arbitrarily Long Arithmetic Progressions of primes

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Abstract

Using Jiang functions $J_2(\omega)$, $J_3(\omega)$ and $J_4(\omega)$ we prove both arbitrarily long arithmetic progressions of primes: (1) $P_{i+1} = P_1^n + di,\quad (P_1, d) = 1$, $i = 1, 2, \cdots, k-1, n \geq 1$, which have the same Jiang function; (2) $P_{i+1} = P_1^n + \omega_g i, i = 1, 2, \cdots, k-1, n \geq 1$, $\omega_g = \prod_{2 \leq P \leq P_g} P$ and generalized arithmetic progressions of primes $P_i = P + i\omega_g$ and $P_{k+i} = P^n + i\omega_g$, $i = 1, \cdots, k, n \geq 2$.

The Green-Tao theorem is false, because they do not prove the twin primes theorem and arithmetic progressions of primes [3].

In prime numbers theory there are both well-known conjectures that there exist both arbitrarily long arithmetic progressions of primes. In this paper using Jiang functions $J_2(\omega)$, $J_3(\omega)$ and $J_4(\omega)$ we obtain the simplest proofs of both arbitrarily long arithmetic progressions of primes.

Theorem 1. We define arithmetic progressions of primes:

$$P_1, P_2 = P_1 + d, P_3 = P_1 + 2d, \cdots, P_k = P_1 + (k-1)d, (P_1, d) = 1. \qquad (1)$$

We rewrite (1)

$$P_3 = 2P_2 - P_1, \quad P_j = (j-1)P_2 - (j-2)P_1, \quad 3 \le j \le k. \qquad （2）$$

We have Jiang function [1]

$$J_3(\omega) = \prod_{3 \le P} \left[(P-1)^2 - X(P) \right], \qquad （3）$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=3}^{k} \left[(j-1)q_2 - (j-2)q_1 \right] \equiv 0 (\text{mod } P), \qquad （4）$$

where $q_1 = 1, 2, \cdots, P-1; q_2 = 1, 2, \cdots, P-1.$

From (4) we have

$$J_3(\omega) = \prod_{3 \le P < k} (P-1) \prod_{k \le P} (P-1)(P-k+1) \to \infty \text{ as } \omega \to \infty. \qquad （5）$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3, \cdots, P_k$ are all primes for all $k \ge 3$. It is a generalization of Euclid and Euler proofs for the existence of infinitely many primes [1].

We have the best asymptotic formula [1]

$$\begin{aligned} \pi_{k-1}(N,3) &= \left| \left\{ (j-1)P_2 - (j-2)P_1 = \text{prime}, 3 \le j \le k, P_1, P_2 \le N \right\} \right| \\ &= \frac{J_3(\omega)\omega^{k-2}}{2\phi^k(\omega)} \frac{N^2}{\log^k N} (1 + o(1)), \end{aligned} \qquad （6）$$

where $\omega = \prod_{2\le P} P, \phi(\omega) = \prod_{2\le P}(P-1)$, （7）

ω is called primorials, $\phi(\omega)$ Euler function.

(6) is a generalization of the prime number theorem $\pi(N) = \frac{N}{\log N}(1+o(1))$ [1].

Substituting (5) and (7) into (6) we have the best asymptotic formula

$$\pi_{k-1}(N,3) = \frac{1}{2}\prod_{2\le P<k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k\le P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \qquad (8)$$

From (8) we are able to find the smallest solution $\pi_{k-1}(N_0,3) > 1$ for large k.

Grosswald and Zagier obtain heuristically even asymptotic formulae [2]. Let $k=2$ and $d=2$. From (1) we have twin primes theorem: $P_2 = P_1 + 2$. The Green-Tao theorem is false, because they do not prove the twin primes theorem and arithmetic progressions of primes [3].

Example 1. Let $k=3$. From (2) we have

$$P_3 = 2P_2 - P_1. \qquad (9)$$

From (5) we have

$$J_3(\omega) = \prod_{3\le P}(P-1)(P-2) \to \infty \text{ as } \omega \to \infty. \qquad (10)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 are primes. From (8) we have the best asymptotic formula

$$\pi_2(N,3) = \prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N^2}{\log^3 N}(1+o(1)) = 0.66016\frac{N^2}{\log^3 N}(1+o(1)). \qquad (11)$$

Example 2. Let $k=4$. From (2) we have

$$P_3 = 2P_2 - P_1, \qquad P_4 = 3P_2 - 2P_1. \qquad (12)$$

From (5) we have

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-3)\to\infty \quad \text{as} \quad \omega\to\infty. \qquad （13）$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 and P_4 are all primes. From (8) we have the best asymptotic formula

$$\pi_3(N,3)=\frac{9}{4}\prod_{5\le P}\frac{P^2(P-3)}{(P-1)^3}\frac{N^2}{\log^4 N}(1+o(1)). \qquad （14）$$

Example 3. Let $k=5$. From (2) we have

$$P_3=2P_2-P_1, \quad P_4=3P_2-2P_1, \qquad P_5=4P_2-3P_1. \qquad （15）$$

From (5) we have

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-4)\to\infty \quad \text{as} \quad \omega\to\infty. \qquad （16）$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3, P_4 and P_5 are all primes. From (8) we have the best asymptotic formula

$$\pi_4(N,3)=\frac{27}{4}\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N^2}{\log^5 N}(1+o(1)). \qquad （17）$$

Theorem 2. From (1) we obtain

$$P_4=P_3+P_2-P_1, \qquad P_j=P_3+(j-3)P_2-(j-3)P_1, \quad 4\le j\le k. \qquad （18）$$

We have Jiang function [1]

$$J_4(\omega)=\prod_{3\le P}((P-1)^3-X(P)), \qquad （19）$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=4}^{k}(q_3+(j-3)q_2-(j-3)q_1)\equiv 0(\mathrm{mod}\ P), \qquad （20）$$

where $q_i=1,2,\cdots,P-1, \quad i=1,2,3.$

From (20) we have

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2\prod_{(k-1)\le P}(P-1)\left[(P-1)^2-(P-2)(k-3)\right]\to\infty$$

as $\omega\to\infty$. (21)

We prove there exist infinitely many primes P_1, P_2 and P_3 such that $P_4,\cdots,P_k$ are all primes for all $k\ge 4$.

We have the best asymptotic formula [1]

$$\pi_{k-2}(N,4)=\left|\left\{P_3+(j-3)P_2-(j-3)P_1=\text{prime},4\le j\le k,P_1,P_2,P_3\le N\right\}\right|$$

$$=\frac{J_4(\omega)\omega^{k-3}}{6\phi^k(\omega)}\frac{N^3}{\log^k N}(1+o(1)).\qquad（22）$$

Substituting (7) and (21) into (22) we have

$$\pi_{k-2}(N,4)$$

$$=\frac{1}{6}\prod_{2\le P<(k-1)}\frac{P^{k-3}}{(P-1)^{k-2}}\prod_{(k-1)\le P}\frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}}\frac{N^3}{\log^k N}(1+o(1)).\qquad(23)$$

From (23) we are able to find the smallest solution $\pi_{k-2}(N_0,4)>1$ for large k.

Example 4. Let $k=4$. From (18) we have

$$P_4=P_3+P_2-P_1\qquad（24）$$

From (21) we have

$$J_4(\omega)=\prod_{3\le P}(P-1)(P^2-3P+3)\to\infty\ \text{ as }\ \omega\to\infty.\qquad（25）$$

We prove there exist infinitely many primes P_1, P_2 and P_3 such that P_4 are primes.From (23) we have

$$\pi_2(N,4)=\frac{1}{3}\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\frac{N^3}{\log^4 N}(1+o(1)).\qquad（26）$$

From (1) We obtain the following equations:

$$\pi_{k-3}(N,5) = \left|\left\{P_4 + (j-3)P_3 - (j-2)P_2 + P_1 = \text{prime}, 5 \le j \le k, P_1, \cdots, P_4 \le N\right\}\right|$$

$$= \frac{1}{24} \frac{J_5(\omega)\omega^{k-4}}{\phi^k(\omega)} \frac{N^4}{\log^k N}(1+o(1)) \tag{27}$$

$$\pi_{k-4}(N,6) = \left|\left\{P_5 + (j-4)P_4 - (j-4)P_3 - P_2 + P_1 = \text{prime}, 6 \le j \le k, P_1, \cdots, P_5 \le N\right\}\right|$$

$$= \frac{1}{120} \frac{J_6(\omega)\omega^{k-5}}{\phi^k(\omega)} \frac{N^5}{\log^k N}(1+o(1)) \tag{28}$$

Theorem 3. We define arithmetic progressions of primes:

$$P_{i+1} = P_1^2 + di, i = 1, 2, \cdots, k-1. \quad \text{（29）}$$

From (29) we have

$$P_3 = 2P_2 - P_1^2, \quad P_j = (j-1)P_2 - (j-2)P_1^2, \quad 3 \le j \le k. \tag{30}$$

We have Jiang function [1]

$$J_3(\omega) = \prod_{3 \le P}\left[(P-1)^2 - X(P)\right], \tag{31}$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=3}^{k}\left[(j-1)q_2 - (j-2)q_1^2\right] \equiv 0 (\text{mod } P), \tag{32}$$

where $q_1 = 1, 2, \cdots, P-1$, $q_2 = 1, 2, \cdots, P-1$.

From (32) we have

$$J_3(\omega) = \prod_{3 \le P < k}(P-1)\prod_{k \le P}(P-1)(P-k+1) \to \infty \quad \text{as} \quad \omega \to \infty. \quad \text{（33）}$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3, \cdots, P_k$ are all primes for all $k \ge 3$. We have the best asymptotic formula [1]

$$\pi_{k-1}(N,3) = \left|\left\{(j-1)P_2 - (j-2)P_1^2 = \text{prime}, 3 \le j \le k, P_1, P_2 \le N\right\}\right|$$

$$= \frac{1}{2^{k-1}} \frac{J_3(\omega)\omega^{k-2}}{\phi^k(\omega)} \frac{N^2}{\log^k N}(1+o(1)). \tag{34}$$

Substituting (7) and (33) into (34) we have

$$\pi_{k-1}(N,3)=\frac{1}{2^{k-1}}\prod_{2\le P<k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k\le P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \tag{35}$$

Theorem 4. We define arithmetic progressions of primes:

$$P_{i+1}=P_1^5+di, i=1,2,\cdots,k-1. \quad （36）$$

From (36) we have

$$P_4=P_3+P_2-P_1^5,\quad P_j=P_3+(j-3)P_2-(j-3)P_1^5,\quad 4\le j\le k. \tag{37}$$

We have Jiang function [1]

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2\prod_{(k-1)\le P}(P-1)\left[(P-1)^2-(P-2)(k-3)\right]\to\infty$$

$$\text{as}\,\omega\to\infty. \tag{38}$$

We prove that there exist infinitely many primes P_1, P_2 and P_3 such that

$P_4,\cdots,P_k$ are all primes for all $k\ge 4$..

We have the best asymptotic formula

$$\pi_{k-2}(N,4)=\left|\left\{P_3+(j-3)P_2-(j-3)P_1^5=\text{prime}, 4\le j\le k, P_1,P_2,P_3\le N\right\}\right|$$
$$=\frac{1}{6\times 5^{k-3}}\frac{J_4(\omega)\omega^{k-3}}{\phi^k(\omega)}\frac{N^3}{\log^k N}(1+o(1)).$$

（39）

Theorem 5. We define arithmetic progressions of primes:

$$P_{j+1}=P_1^n+di, i=1,2,\cdots,k-1, n\ge 1. \quad （40）$$

From (40) we have

$$P_3=2P_2-P_1^n,\quad P_j=(j-1)P_2-(j-2)P_1^n. \tag{41}$$

We have Jiang function [1]

$$J_3(\omega)=\prod_{3\le P<k}(P-1)\prod_{k\le P}(P-1)(P-k+1)\to\infty\quad\text{as}\quad\omega\to\infty. \quad （42）$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3,\cdots,P_k$ are all primes for all $k \geq 3$.

We have the best asymptotic formula [1]

$$\pi_{k-1}(N,3)=\left|\left\{(j-1)P_2-(j-2)P_1^n=\text{prime},3\leq j\leq k,P_1,P_2\leq N\right\}\right|$$

$$=\frac{1}{2\times n^{k-2}}\frac{J_3(\omega)\omega^{k-2}}{\phi^k(\omega)}\frac{N^2}{\log^k N}. \quad （43）$$

Substituting (7) and (42) into (43) we have

$$\pi_{k-1}(N,3)=\frac{1}{2\times n^{k-2}}\prod_{2\leq P<k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k\leq P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \quad （44）$$

Theorem 6. We define arithmetic progressions of primes:

$$P_{j+1}=P_1^n+di,i=1,2,\cdots,k-1,n\geq 1. \quad （45）$$

From (45) we have

$$P_4=P_3+P_2-P_1^n,\quad P_j=P_3+(j-3)P_2-(j-3)P_1^n,\quad 4\leq j\leq k. \quad (46)$$

We have Jiang function [1]

$$J_4(\omega)=\prod_{3\leq P<(k-1)}(P-1)^2\prod_{(k-1)\leq P}(P-1)\left[(P-1)^2-(P-2)(k-3)\right]\rightarrow\infty$$

as $\omega\rightarrow\infty$. (47)

We prove that there exist infinitely many primes P_1, P_2 and P_3 such that $P_4,\cdots,P_k$ are all primes for all $k\geq 4$..

We have the best asymptotic formula [1]

$$\pi_{k-2}(N,4)=\left|\left\{P_3+(j-3)P_2-(j-3)P_1^n=\text{prime},4\leq j\leq k,P_1,P_2,P_3\leq N\right\}\right|$$

$$=\frac{1}{6\times n^{k-3}}\frac{J_4(\omega)\omega^{k-3}}{\phi^k(\omega)}\frac{N^3}{\log^k N}(1+o(1)). \quad （48）$$

Substituting (7) and (47) into (48) we have

$$\pi_{k-2}(N,4)$$
$$=\frac{1}{6\times n^{k-3}}\prod_{2\le P<(k-1)}\frac{P^{k-3}}{(P-1)^{k-2}}\prod_{(k-1)\le P}\frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}}\frac{N^3}{\log^k N}(1+o(1)).$$

(49)

Theorem 7. We define another arithmetic progressions of primes [1, 4]:

$$P_{i+1}=P_1+\omega_g i, i=1,2,\cdots,k-1 \qquad （50）$$

where $\omega_g=\prod_{2\le P\le P_g}$ is called a common difference, P_g is called $g-$th prime.

We have Jiang function [1, 4]

$$J_2(\omega)=\prod_{3\le P}(P-1-X(P)), \qquad （51）$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k-1}(q+\omega_g i)\equiv 0(\mathrm{mod}\ P), \qquad （52）$$

where $q=1,2,\cdots,P-1$.

If $P\big|\omega_g$, then $X(P)=0$; $X(P)=k-1$ otherwise. From (52) we have

$$J_2(\omega)=\prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-k). \qquad （53）$$

If $k=P_{g+1}$ then $J_2(P_{g+1})=0$, $J_2(\omega)=0$, there exist finite primes P_1 such that $P_2,\cdots,P_k$ are all primes. If $k<P_{g+1}$ then $J_2(\omega)\ne 0$, there exist infinitely many primes P_1 such that $P_2,\cdots,P_k$ are all primes. We have the best asymptotic formula [1,4]

$$\pi_k(N,2)=\left|\left\{P_1+\omega_g i=\text{prime},1\le i\le k-1,P_1\le N\right\}\right|$$
$$=\frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}(1+o(1)). \tag{54}$$

Let $k=P_{g+1}-1$. From (50) we have

$$P_{i+1}=P_1+\omega_g i, i=1,2,\cdots,P_{g+1}-2. \tag{55}$$

From (53) we have [1, 4]

$$J_2(\omega)=\prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-P_{g+1}+1)\to\infty \quad \text{as} \quad \omega\to\infty \tag{56}$$

We prove that there exist infinitely many primes P_1 such that $P_2,\cdots,P_{P_{g+1}-1}$ are all primes for all P_{g+1}.

Substituting (7) and (56) into (54) we have

$$\pi_{P_{g+1}-1}(N,2)=$$
$$\prod_{2\le P\le P_g}\left(\frac{P}{P-1}\right)^{P_{g+1}-2}\prod_{P_{g+1}\le P}=\frac{P^{P_{g+1}-2}(P-P_{g+1}+1)}{(P-1)^{P_{g+1}-1}}\frac{N}{(\log N)^{P_{g+1}-1}}(1+o(1)). \tag{57}$$

From (57) we are able to find the smallest solutions $\pi_{P_{g+1}-1}(N_0,2)>1$ for large P_{g+1}.

Example 5. Let $P_1=2$, $\omega_1=2$, $P_2=3$. From (55) we have the twin primes theorem

$$P_2=P_1+2. \tag{58}$$

From (56) we have

$$J_2(\omega)=\prod_{3\le P}(P-2)\to\infty \quad \text{as} \quad \omega\to\infty, \tag{59}$$

We prove that there exist infinitely many primes P_1 such that P_2 are primes. From

(57) we have the best asymptotic formula

$$\pi_2(N,2)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N}{\log^2 N}(1+o(1)). \qquad (60)$$

Example 6. Let $P_2=3$, $\omega_2=6$, $P_3=5$. From (55) we have

$$P_{i+1}=P_1+6i, i=1,2,3. \qquad (61)$$

From (56) we have

$$J_2(\omega)=2\prod_{5\le P}(P-4)\to\infty \text{ as } \omega\to\infty. \qquad (62)$$

We prove that there exist infinitely many primes P_1 such that P_2, P_3 and P_4 are all primes. From (57) we have the best asymptotic formula

$$\pi_4(N,2)=27\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N}{\log^4 N}(1+o(1)). \qquad (63)$$

Example 7. Let $P_9=23$, $\omega_9=223092870$, $P_{10}=29$. From (55) we have

$$P_{i+1}=P_1+223092870i, i=1,2,\cdots,27. \qquad (64)$$

From (56) we have

$$J_2(\omega)=36495360\prod_{29\le P}(P-28)\to\infty \text{ as } \omega\to\infty \qquad (65)$$

We prove that there exist infinitely many primes P_1 such that $P_2,\cdots,P_{28}$ are all primes. From (57) we have the best asymptotic formula

$$\pi_{28}(N,2)=\prod_{2\le P\le 23}\left(\frac{P}{P-1}\right)^{27}\prod_{29\le P}\frac{P^{27}(P-28)}{(P-1)^{28}}\frac{N}{\log^{28} N}(1+o(1)). \qquad (66)$$

From (66) we are able to find the smallest solutions $\pi_{28}(N_0,2)>1$.

Theorem 8. We define another arithmetic progressions of primes:

$$P_{i+1}=P_1^n+\omega_g i, i=1,2,\cdots,k-1, n\ge 1. \qquad (67)$$

We have Jiang function [1]

$$J_2(\omega)=\prod_{3\le P}(P-1-X(P)), \qquad (68)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k-1}(q_1^n+\omega_g i)\equiv 0(\bmod P), \qquad (69)$$

where $q_1=1,2,\cdots P-1$.

If $X(P)=P-1$ and $J_2(P)=0$, then there exist finite primes P_1 such that $P_2,\cdots P_k$ are primes. If $X(P)<P-1$ and $J_2(\omega)\ne 0$, then there exist infinitely many primes P_1 such that $P_2,\cdots,P_k$ are all prime for all P_g.

We have the best asymptotic formula [1]

$$\begin{aligned}\pi_k(N,2)&=\left|\left\{P_1^n+\omega_g i=\text{prime}, 1\le i\le k-1, P_1\le N\right\}\right|\\&=\frac{1}{n^{k-1}}\frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}(1+o(1)).\end{aligned} \qquad (70)$$

Example 8. Let $n=2$, $k=3$ and $\omega_g=6$. From (67) we have

$$P_2=P_1^2+6,\quad P_3=P_1^2+12,\quad P_4=P_1^2+18 \qquad (71)$$

We have Jiang function [1]

$$J_2(\omega)=2\prod_{5\le P}\left(P-4-\left(\frac{-6}{P}\right)-\left(\frac{-3}{P}\right)-\left(\frac{-2}{P}\right)\right)\to\infty \text{ as } \omega\to\infty \qquad (72)$$

where $\left(\frac{-6}{P}\right),\left(\frac{-3}{P}\right)$ and $\left(\frac{-2}{P}\right)$ denote the Legendre symbols.

We prove that there exist infinitely many primes P_1 such that P_2, P_3 and P_4 are all primes. We have the best asymptotic formula [1]

$$\pi_4(N,2)=\left|\left\{P_1^2+6i=\text{prime}, i=1,2,3, P_1\le N\right\}\right| \\ =\frac{1}{8}\frac{J_2(\omega)\omega^3}{\phi^4(\omega)}\frac{N}{\log^4 N}(1+o(1)). \tag{73}$$

We shall move on to the study of the generalized arithmetic progression of consecutive primes [5]. A generalized arithmetic progression of consecutive primes is defined to be the sequence of primes,

$$P, P+\omega_g, P+2\omega_g, \cdots, P+k\omega_g \text{ and } P^n+\omega_g, P^n+2\omega_g, \cdots, P^n+k\omega_g,$$

where P is the first term, $n\ge 2$. For example, 5, 11, 17, 23, and 31, 37, 43, is a generalized arithmetic progression of primes with $P=5$, $\omega_g=6$, $k=3$ and $n=2$.

Theorem 9. We define the generalized arithmetic progressions:

$$P_i=P+i\omega_g \text{ and } P_{k+i}=P^n+i\omega_g \tag{74}$$

where $i=1,\cdots,k, n\ge 2$.

We have Jiang function [1]

$$J_2(\omega)=\prod_{3\le P}\left(P-1-X(p)\right), \tag{75}$$

$X(P)$ is the number of solutions of congruence

$$\prod_{i=1}^{k}(q+i\omega_g)(q^n+i\omega_g)\equiv 0(\bmod P), \tag{76}$$

$q=1,2,\cdots P-1.$

If $X(P)=P-1$ and $J_2(P)=0$, then there exist finite primes P such that $P_1, P_2, \cdots, P_{2k}$ are primes. If $X(P)<P-1$, $J_2(\omega)\ne 0$, then there exist infinitely

many primes P such that $P_1, P_2, \cdots, P_{2k}$ are all primes.

If $J_2(\omega) \neq 0$, we have the best asymptotic formula of the number of primes $P \leq N$ [1]

$$\pi_{2k+1}(N,2) = \frac{J_2(\omega)\omega^{2k}}{n^k \phi^{2k+1}(\omega)} \frac{N}{(\log N)^{2k+1}} (1+o(1)). \tag{77}$$

Example 9. Let $\omega_g = 6, k = 3$, and $n = 2$. From (74) we have

$$P_1 = P+6, P_2 = P+12, P_3 = P+18 \quad \text{and}$$

$$P_4 = P^2+6, P_5 = P^2+12, P_6 = P^2+18. \tag{78}$$

We have Jiang function [1]

$$J_2(\omega) = 12672 \prod_{23 \leq P} \left(P - 7 - \left(\frac{-2}{P} \right) - \left(\frac{-3}{P} \right) - \left(\frac{-6}{P} \right) \right) \neq 0 \tag{79}$$

Since $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many primes P such that $P_1, \cdots, P_6$ are all primes.

From (77) we have

$$\pi_7(N,2) = \frac{J_2(\omega)\omega^6}{8\phi^7(\omega)} \frac{N}{\log^7 N} (1+o(1)). \tag{80}$$

Remark. Theorems 1, 3 and 5 have the same Jiang function $J_3(\omega)$ and theorems 2, 4 and 6 the same Jiang function $J_4(\omega)$ which have the same character. All irreducible prime equations have the Jiang functions and the best asymptotic formulas [1]. In our theory there are no almost primes, for example $P_1 = P_2 P_3 + 2$ and

$N = P_1 + P_2 P_3$ are theorems of three genuine primes. Using the sieve method, circle method, ergodic theory, harmonic analysis, discrete geometry, and combinatories they are not able to attack twin primes conjecture, Goldbach conjecture, long arithmetic progressions of primes and other problems of primes and to find the best asymptotic formulas. The proofs of Szemerédic's theorem are false, because they do not prove the twin primes theorem and arithmetic progressions of primes [3, 6-10].

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