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PROOF OF SCHINZEL'S HYPOTHESIS

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Abstract

In this paper using the arithmetic function $J_2(\omega)$ we prove Schinzel's hypothesis, twin prime theorem and Goldbach theorem. If $J_2(\omega) \neq 0$ but $\pi_2(P,2) = 0$, then there are the finite prime solutions. Using this theorem we prove that there are the finite Fermat's primes, finite Mersenne primes, finite prime repunits, finite Santilli's primes and finite Weiss's primes. Using the arithmetic function $J_2(\omega)$ we prove some theorems which as yet mathematicians cannot even imagine.

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1. Introduction

The Schinzel's hypothesis is a famous open problem. Schinzel[1] asserts that for every collection of irreducible nonconstant polynomials $f_1(x),\cdots,f_k(x)$ with integral coefficients and positive leading coefficients, if there is no fixed integer greater than 1 dividing the product $f_1(m),\cdots,f_k(m)$ for all integers *m*, then there are infinitely many integers *m* such that each of the numbers $f_1(m),\cdots,f_k(m)$ is prime. The case when each of the polynomials is linear was previously conjectured by L. E. Dickson and is known as the prime *k*-tuples conjecture. In this paper using the arithmetic function $J_2(\omega)$ we prove the Schinzel's hypothesis. If $J_2(\omega)\neq 0$, then there exist infinitely many primes *P* such that each of $f_j(P)$ is a prime. If $J_2(\omega)=0$, then there exist the finite prime solutions. If $J_2(\omega)\neq 0$ but $\pi_2(P,2)=0$, then there are the finite prime solutions. Using this theorem we prove that there are the finite Fermat's primes, finite Mersenne primes, finite prime repunits the finite Santilli's primes and finite Weiss's primes. In the same way we can prove that there are the finite primes of these forms: $5\times 2^n\pm 1$, $7\times 2^n\pm 1,\cdots,10^n\pm 3$, $10^n\pm 9$, $2\times 10^n-1$, $2\times 10^n\pm 3$, $2\times 10^n-7$, $2\times 10^n\pm 9$, $3\times 10^n\pm 1$, $3\times 10^n\pm 7$, $4\times 10^n+1$, $4\times 10^n\pm 3$, $5\times 10^n-1$, $5\times 10^n\pm 3$, $5\times 10^n-7$, $5\times 10^n\pm 9$, $6\times 10^n\pm 1$, $6\times 10^n\pm 7$, $7\times 10^n+1$, $7\times 10^n\pm 3$, $7\times 10^n\pm 9$, $8\times 10^n-1$, $8\times 10^n\pm 3$, $8\times 10^n-7$, $8\times 10^n\pm 9$, $9\times 10^n\pm 1$, $9\times 10^n\pm 7$, which as yet mathematicians cannot even imagine.

2. Proof of Schinzel's Hypothesis

Schinzel's Theorem[1]. If there exist infinitely many primes *P* such that each of $f_j(P)$ (for $j=1,\cdots,k-1$) is also a prime, then $f_j(P)$ must satisfy three necessary and sufficient conditions:

（I）Let $f_j(P)$ be $k-1$ distinct polynomials with integral coefficients irreducible over the integers.

（II）There exists an arithmetic function [2-5]

$$J_2(\omega)=\prod_{2\le P\le P_i}(P-1-H(P)),\qquad (1)$$

where $\omega=\prod_{2\le P\le P_i}P$ is called primorials.

Let *H*(*P*) denote the number of solutions of congruence

$$\prod_{j=1}^{k-1} f_j(q) \equiv 0 (\bmod P), \tag{2}$$

where $q = 1, \cdots, P-1$.

If $J_2(\omega) \neq 0$, then there exist infinitely many primes P such that each of $f_j(P)$ is a prime .If $J_2(\omega) = 0$, then there exist the finite prime solutions. It is a generalization of Euler proof of the existence of infinitely many primes.

（III）We have the best asymptotic formula of the number of primes *P* less than *N* [2-5],

$$\pi_k(N,2) \sim \prod_{j=1}^{k-1} (\deg f_j)^{-1} \times \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}, \tag{3}$$

where $\phi(\omega) = \prod_{2 \le P \le P_i} (P-1)$ is called Euler function of primorials.

3. Applications of Schinzel's Theorem

Using the Schinzel's theorem we prove the following prime theorems.

Theorem 1. The prime 3-tuples, $P + b : b = 0,2,4$.

From (2) we have the arithmetic function

$$J_2(3) = 0. \tag{4}$$

Therefore there are no prime 3-tuples if $P \neq 3$.

Theorem 2. The prime 5-tuples, $P + b : b = 0, 2, 6, 8, 14$.

From (2) we have

$$J_2(5) = 0. \tag{5}$$

Therefore there are no prime 5-tuples if $P \neq 5$.

Theorem 3. The prime 7-tuples, $P + b : b = 0,4,6,10,12,16,22.$

From (2) we have

$$J_2(7) = 0. \tag{6}$$

Therefore there are no prime 7-tuples if $P \neq 7$.

Theorem 4. The prime 11-tuples, $P + b : b = 0,2,6,8,12,18,20,26,32,36,60.$

Frome (2) we have

$$J_2(11) = 0. \tag{7}$$

Therefore there are no prime 11-tuples if $P \neq 11$.

Theorem 5. Twin prime theorem, $P_1 = P + 2$.

From (2) we have

$$J_2(\omega) = \prod_{3 \le P \le P_i} (P - 2) \neq 0. \tag{8}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes P such that P_1 is a primes.

From (3) we have the best asymptotic formula of the number of primes P less than N

$$\pi_2(N,2) \sim 2 \prod_{3 \le P \le P_i} \left(1 - \frac{1}{(P-1)^2}\right) \frac{N}{\log^2 N}. \tag{9}$$

Theorem 6. $P_1 = P + 2$ and $P_2 = P + 6$.

From (2) we have

$$J_2(\omega) = \prod_{5 \le P \le P_i} (P - 3) \neq 0. \tag{10}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes P such that P_1 and P_2 are primes.

From (3) we have

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{\phi^2(\omega)} \frac{N}{\log^3 N}. \tag{11}$$

Theorem 7. Goldbach theorem, $N = P_1 + P_2$.

From (2) we have

$$J_2(\omega) = \prod_{3 \le P \le P_i} (P - 2) \prod_{P|N} \frac{P-1}{P-2} \neq 0. \tag{12}$$

Since $J_2(\omega) \neq 0$, every even number N greater than 4 is the sum of two odd primes.

From (3) we have

$$\pi_2(N,2) \sim 2 \prod_{3 \le P \le P_i} \left(1 - \frac{1}{(P-1)^2}\right) \prod_{P|N} \frac{P-1}{P-2} \frac{N}{\log^2 N}. \tag{13}$$

Theorme 8. $P_1 = P + 6$ and $P_2 = N - P$.

From (2) we have

$$J_2(\omega) = \prod_{3|N} (P-1) \prod_{5 \le P \le P_i} (P-3) \prod_{P|N, P|(N+6)} \frac{P-2}{P-3} \ne 0 . \tag{14}$$

Since $J_2(\omega) \ne 0$, there exist infinitely many primes *P* such that P_1 and P_2 are primes as $N \to \infty$.

Frome (3) we have

$$\pi_3(N,2) \sim \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N} . \tag{15}$$

Theorem 9. There are the finite Fermat's primes.

Proof. Suppose that

$$P_1 = (P-1)^{2^n} + 1 . \tag{16}$$

From (2) we have

$$J_2(\omega) = \prod_{3 \le P \le P_i} (P - 1 - \chi(P)) \ne 0, \tag{17}$$

where $\chi(P) = 2^n$ if $P \equiv 1 (\bmod 2^{n+1})$; $\chi(p) = 0$ otherwise.

Since $J_2(\omega) \ne 0$, there exist infinitely many primes *P* such that P_1 is a prime for every integer *n*.

From (3) we have

$$\pi_2(N,2) \sim \frac{1}{2^n} \frac{J_2(\omega)\omega}{\phi^2(\omega)} \frac{N}{\log^2 N} . \tag{18}$$

When *P*=3, numbers $P_1 = 2^{2^n} + 1$ of this form are called Fermat's numbers, and primes of this form are called Fermat's primes. From (18) we have $\pi_2(3,2) \to 0$ as $n \to \infty$. We prove that there are the finite Fermat's primes.

Theorem 10. There are the finite Mersenne primes and finite prime repunits.

Proof. Suppose that

$$P_1 = \frac{(P-1)^{P_0} - 1}{P-2}, \tag{19}$$

where P_0 is an odd prime.

From (2) we have

$$J_2(\omega)=\prod_{3\le P\le P_i}(P-1-\chi(P))\neq 0, \tag{20}$$

where $\chi(P_0)=1$, $\chi(P)=P_0-1$ if $P\equiv 1(\bmod P_0)$; $\chi(P)=0$ otherwise.
Since $J_2(\omega)\neq 0$ there exist infinitely many primes P such that P_1 is a prime.

From (3) we have

$$\pi_2(N,2)\sim\frac{1}{P_0-1}\frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}. \tag{21}$$

When P=3, numbers $P_1=2^{P_0}-1$ of this form are called Mersenne numbers, and primes of this form are called Mersenne primes. From (21) we have $\pi_2(3,2)\to 0$ as $P_0\to\infty$. We prove that there are the finite Mersenne primes. When P=11, numbers $P_1=\frac{10^{P_0}-1}{9}$ of this form are called repunits, and primes of this form are called prime repunits. From (21) we have $\pi_2(11,2)\to 0$ as $P_0\to\infty$. We prove that there are the finite prime repunits

Theorem 11. There are the finite Santilli's primes.

Proof. Suppose that

$$P_1=\frac{(P-1)^{P_0}+1}{P}, \tag{22}$$

where P_0 is an odd prime.

From (2) we have

$$J_2(\omega)=\prod_{3\le P\le P_i}(P-1-\chi(P))\neq 0, \tag{23}$$

where $\chi(P)=P_0-1$ if $P\equiv 1(\bmod P_0)$; $\chi(P)=0$ otherwise.
Since $J_2(\omega)\neq 0$, there exist infinitely many primes P such that P_1 is a prime.

From (3) we have

$$\pi_2(N,2)\sim\frac{1}{P_0-1}\frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}\ . \tag{24}$$

When P=3, numbers $P_1=\frac{2^{P_0}+1}{3}$ of this form are called the Santilli's numbers, and primes of this form are called the Santilli's primes. From (24) we have $\pi_2(3,2) \to 0$ as $P_0 \to \infty$. We prove that there are the finite Santilli's primes. When *P*=11, numbers $P_1 = \frac{10^{P_0}+1}{11}$ of this form are called the Santilli's numbers, and primes of this form are called the Santilli's primes. From (24) we have $\pi_2(11,2) \to 0$ as $P_0 \to \infty$. We prove that there are the finite Santilli's primes.

Theorem 12. There are the finite Weiss's primes.

Proof. Suppose that

$$P_1 = 3 \times (P-1)^n \pm 1, \tag{25}$$

where *n* is an integer.

From (2) we have

$$J_2(\omega) = \prod_{3 \le P \le P_i} (P - 1 - H(P)) \ne 0, \tag{26}$$

where $H(p)$ is the number of solutions of congruence

$$3 \times (q-1)^n \pm 1 \equiv 0 (\bmod P), \tag{27}$$

$q = 1, \cdots, P-1.$

Since $J_2(\omega) \ne 0$, there exist infinitely many primes *P* such that P_1 is a prime.

From (3) we have

$$\pi_2(N,2) \sim \frac{J_2(\omega)\omega}{n\phi^2(\omega)} \frac{N}{\log^2 N}. \tag{28}$$

When *P*=3, numbers $P_1 = 3 \times 2^n \pm 1$ of this form are called Weiss's numbers, and primes of this form are called Weiss's primes. From (28) we have that $\pi_2(3,2) \to 0$ as $n \to \infty$. We prove that there are finite Weiss's primes.

Theorem 13. $P_1 = 5P^2 + 6, P_2 = 25P^2 + 36, P_3 = 125P^2 + 216$.

From (2) we have

$$J_2(\omega) = 384 \prod_{13 \le P \le P_i} (P - 4 - 2(\frac{-30}{P}) - (\frac{-1}{P})) \ne 0. \tag{29}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes P such that P_1, P_2 and P_3 are primes.

From (3) we have

$$\pi_4(N,2) \sim \frac{J_2(\omega)\omega^3}{8\phi^4(\omega)} \frac{N}{\log^4 N}. \tag{30}$$

Theorem 14. $P_1 = P^2 + 6, P_2 = P^2 + 12, P_3 = P^2 + 18$

From (2) we have

$$J_2(\omega) = 2 \prod_{5 \le P \le P_i} \left(P - 4 - \left(\frac{-6}{P}\right) - \left(\frac{-3}{P}\right) - \left(\frac{-2}{P}\right) \right) \neq 0. \tag{31}$$

Since $J_2(\omega) \neq 0$, there exist infinitely many primes P such that P_1, P_2 and P_3 are primes.

From (3) we have

$$\pi_4(N,2) \sim \frac{J_2(\omega)\omega^3}{8\phi^4(\omega)} \frac{N}{\log^4 N}. \tag{32}$$

Theorem 14. $P_j = 2^j(P^2 - 1) + 1$ for $j = 1, \cdots, k-1$.

From (2) we have

$$J_2(\omega) = \prod_{3 \le P \le P_i} \left(P - n - \sum_{j=1}^{n-1} \left(\frac{2^j(2^j - 1)}{P} \right) \right) \neq 0. \tag{33}$$

We define the smallest positive integer s such that

$$2^s \equiv 1 (\bmod P). \tag{34}$$

We have $n = k$ if $k < s$; $n = s$ if $k \ge s$.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes P such that P_j are primes for any length k.

From (3) we have

$$\pi_k(N,2) \sim \frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)} \frac{N}{\log^k N}. \tag{35}$$

Theorem 15. Suppose that $P_j = 3^j(P^2 - 1) + 1$ for $j = 1, \cdots, k-1$.

From (2) we have

$$J_2(\omega) = 2\prod_{5\le P\le P_i}\left(P-n-\sum_{j=1}^{n-1}\left(\frac{3^j(3^j-1)}{P}\right)\right)\ne 0. \tag{36}$$

We define the smallest positive integer *s* such that

$$3^s \equiv 1(\bmod P). \tag{37}$$

We have $n=k$ if $k<s$; $n=s$ if $k\ge s$.

Since $J_2(w)\ne 0$, there exist infinitely many primes *P* such that P_j are primes for any length *k*.

From (3) we have

$$\pi_k(N,2)\sim\frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{38}$$

Theorem 16. Let $P_j = m^j(P^2-1)+1$ for $j=1,\cdots,k-1, m>1$.

From (2) we have

$$J_2(\omega) = \prod_{3\le P\le P_i}\left(P-n-\sum_{j=1}^{n-1}\left(\frac{m^j(m^j-1)}{P}\right)\right)\ne 0. \tag{39}$$

We define the smallest positive integer *s* such that

$$m^s \equiv 1(\bmod P). \tag{40}$$

We have $n=k$ if $k<s$; $n=s$ if $k\ge s$; $J_2(P)=P-1$ if $P\mid m(m-1)$.

Since $J_2(\omega)\ne 0$, there exist infinitely many primes *P* such that P_j are primes for any length *k*.

From (3) we have

$$\pi_k(N,2)\sim\frac{J_2(\omega)\omega^{k-1}}{2^{k-1}\phi^k(\omega)}\frac{N}{\log^k N}. \tag{41}$$

References

[1] A. Schinzel and W. Sierpinski, Sur certaines hypotheses concernant les nombres premiers. Acta Arith. 4(1958) 185-208.

[2] C. X. Jiang. Foundations of Santilli’s isonumber theory 1. Algebras, Groups and Geometries 15(1998)351-393. MR 2000c: 11214.

[3] C. X. Jiang. Foundations of Santilli's isonumber theory 2. Algebras, Groups and Geometries 15(1998) 509-544.

[4] C. X. Jiang. Foundations of Santilli's isonumber theory. In: Foundamental open problems in sciences at the end of the millennium, T. Gill, K. Liu and E. Trell(Eds) Hadronic Press, USA, 105-139(1999).

[5] C. X. Jiang. Foundations of Santilli's isonumber theory. Hadronic Press, to appear in 2001. Using the arithmetic functions $J_n(\omega)$ the Author proves about 600 prime theorems. This monograph will open up a new world of mathematics which as yet mathematicians cannot even imagine.