An extension of Dirac notation and its consequences

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In this article we observe first that there is an extension of Dirac notation to include the so called "external interchangeability" of the hermitian operators when taking the real part of the bracket in the bra-ket notation. Then we conclude that this, together with the usual "internal interchangeability" of the hermitian operators concerning the real part of the bracket leads to the commutability of operators as to both cross and dot product therefore (by making use of the fact that operators are linear) all the vector algebra is valid when taking the real part of the bracket and we have hermitian operators. We deal with one particle situations and bound states. Finally we find a new way for calculating the square of the Runge-Lenz vector and we make speculations about hidden variable. First principles are used for the deduction of our conclusions,like the independence of the phase angle of ψ from its absolute value and the commutation relationships which are valid for any function. Integration as to volume is meant from zero to infinity and units are omitted for ease of calculations.

I. INTERCHANGEABILITY OF HERMITIAN OPERATORS

Let us begin by giving some definitions. The so called in this article "external interchangeability" that is to be proved is the following:

$$\text{(1.1)} \qquad \operatorname{Re}\left[\left\langle -\hat{B}\psi \mid \times \mid \hat{A}\psi \right\rangle\right] = \operatorname{Re}\left[\left\langle \psi \mid \hat{A}\times\hat{B} \mid \psi \right\rangle\right]$$

The so called, on the other hand "internal interchangeability" is:

$$\text{(1.2)} \qquad \operatorname{Re}\langle \psi \mid AB \mid \psi \rangle = \operatorname{Re}\langle B\psi \mid A\psi \rangle$$

Both are valid when A,B are hermitian.

We are going to start by proving that the operator AB-BA is antihermitian when A,B are hermitian. First we notice that :

$$\begin{aligned} &\langle \psi \mid AB - BA \mid \psi \rangle = \langle A\psi \mid B\psi \rangle - \langle B\psi \mid A\psi \rangle = \\ &= \langle A\psi \mid B\psi \rangle - \langle A\psi \mid B\psi \rangle^* \end{aligned} \qquad (1.3)$$

The last part of the equality containing the two terms is obviously purely imaginary, thus:

$$\text{(1.4)} \qquad \operatorname{Re}\langle \psi \mid AB - BA \mid \psi \rangle = 0$$

The validity of identity (1.4) by taking in mind that A,B are hermitian and can be interchanged readily proves (1.2) that is the internal interchangeability of the operators concerning the real part of the bracket.
Next we are going to prove that :

(1.5) $$\mathrm{Re}\Big[\Big\langle -\hat{B}\psi \,|\times|\, \hat{A}\psi\Big\rangle\Big] = \mathrm{Re}\Big[\Big\langle \psi \,|\, \hat{A}\times\hat{B} \,|\, \psi\Big\rangle\Big] = \mathrm{Re}(\vec{C})$$

If (1.5) is to be valid obviously then we also have:

(1.6) $$\mathrm{Re}\Big[-\big\langle \hat{B}\psi \,|\times|\, \hat{A}\psi\big\rangle - \big\langle \psi \,|\, \hat{A}\times\hat{B} \,|\, \psi\big\rangle\Big] = 0$$

For proving (1.5) we decompose into cartesian components. The x component of (1.5) is written as:
(1.7)

$$\mathrm{Re}(C_x) = -\mathrm{Re}\Big[\big\langle B_y\psi \,|\, A_z\psi\big\rangle - \big\langle B_z\psi \,|\, A_y\psi\big\rangle\Big] =$$
$$= -\mathrm{Re}\Big[\big\langle \psi \,|\, B_y A_z \,|\, \psi\big\rangle - \big\langle \psi \,|\, B_z A_y \,|\, \psi\big\rangle\Big] = -\mathrm{Re}\Big[\big\langle \psi \,|\, B_y A_z - B_z A_y \,|\, \psi\big\rangle\Big]$$

The vector operators A,B are hermitian and so are their components. Therefore the x component of the last part of (1.7) is written as:

(1.8) $$\mathrm{Re}(C_x) = -\mathrm{Re}\Big[\big\langle \psi \,|\, B_y A_z - B_z A_y \,|\, \psi\big\rangle\Big] = \mathrm{Re}\Big[\big\langle \psi \,|\, A_y B_z - A_z B_y \,|\, \psi\big\rangle\Big]$$

By subtracting the second part of (1.8) from the second (out of the three) part of (1.7) we get:

(1.9) $$\begin{aligned}&\mathrm{Re}[C_x] = \mathrm{Re}[<\psi \,|\, -B_y A_z + B_z A_y - A_y B_z + A_z B_y \,|\, \psi>] =\\ &= \mathrm{Re}[<\psi \,|\, (B_z A_y - A_y B_z) - (B_y A_z - A_z B_y) \,|\, \psi>]\end{aligned}$$

Both terms in the bracket are of the form AB-BA and therefore are antihermitian with zero real part which concludes the proof of (1.6) since from symmetry the rest of the components are going to obey the same laws.

II. DECOMPOSING INTO HERMITIAN AND ANTIHERMITIAN PARTS – THE ANTI COMMUTATIVITY OF OPERATORS

By using the identity (A.1) proved in the appendix and equation (1.6) we obtain the following identity where we have used the notation that:
$\vec{A} = \hat{A}\psi, \vec{B} = \hat{B}\psi$

(2.1)

$$\mathrm{Re}\left\langle \psi \mid \hat{A}\times\hat{B}-\hat{B}\times\hat{A} \mid \psi \right\rangle = -\mathrm{Re}\left\langle \hat{B}\psi \mid \times \mid \hat{A}\psi \right\rangle + \mathrm{Re}\left\langle \hat{A}\psi \mid \times \mid \hat{B}\psi \right\rangle =$$

$$= \mathrm{Re}\int(-\vec{B}^{*}\times\vec{A}+\vec{A}^{*}\times\vec{B})dV = 2\,\mathrm{Re}\int\vec{A}^{*}\times\vec{B}dV = -2\,\mathrm{Re}\left\langle \psi \mid \hat{B}\times\hat{A} \mid \psi \right\rangle$$

From equating the first and last part of (2.1) we conclude that:

(2.2) $$\mathrm{Re}\left[\left\langle \psi \mid \hat{A}\times\hat{B} \mid \psi \right\rangle\right] = -\mathrm{Re}\left[\left\langle \psi \mid \hat{B}\times\hat{A} \mid \psi \right\rangle\right]$$

So far the so called "anti commutability" of hermitian operators as to the cross product and the real part of the bracket is readily proved. Similarly we find that:

(2.3) $\mathrm{Re} < \psi \mid AB \mid \psi > = \mathrm{Re} < A\psi \mid B\psi >$

And

(2.4) $\mathrm{Re} < \psi \mid BA \mid \psi > = \mathrm{Re} < B\psi \mid A\psi >$

Since from (1.4) we may equate (2.4) and (2.3) we arrive at:

(2.5)

$$\mathrm{Re} < A\psi \mid B\psi > = \mathrm{Re} < B\psi \mid A\psi > \Rightarrow \mathrm{Re} < \psi \mid AB \mid \psi > = \mathrm{Re} < \psi \mid BA \mid \psi >$$

Thus we have proved a similar identity for the dot product. By the way we also find that the operators $\hat{A}\times\hat{B}+\hat{B}\times\hat{A}$ and AB-BA are antihermitian, that is they have no real part.

The operator AB+BA is hermitian if A,B are hermitian. Proof:

Due to hermitian property of A,B

(2.6) $< A\psi \mid B\psi > = < \psi \mid AB \mid \psi >, < B\psi \mid A\psi > = < \psi \mid BA \mid \psi >$

By observing that:

(2.7) $< A\psi \mid B\psi > = < B\psi \mid A\psi >$

and adding up the terms we get:

(2.8)
$$< \psi \mid AB \mid \psi > + < \psi \mid BA \mid \psi > = < \psi \mid AB+BA \mid \psi > =$$
$$= < A\psi \mid B\psi > + < A\psi \mid B\psi >^{*} = real$$

Furthermore the operator $\hat{C}=\hat{A}\times\hat{B}-\hat{B}\times\hat{A}$ is hermitian if A,B are hermitian . Proof:

Its x component is written as

$C_x=(A_yB_z-A_zB_y)-(B_yA_z-B_zA_y)$ (2.9)

$= (A_yB_z+B_zA_y)-(A_zB_y+B_yA_z)$

As can be seen C_x has the form (KL+LK)-(MN+NM) where K,L,M,N, are hermitian operators and thus the operator C is hermitian by making use of what we have proved so far (see 2.8).

We finally may decompose:

(2.10) $$\hat{A}\times\hat{B}=\left(\frac{\hat{A}\times\hat{B}-\hat{B}\times\hat{A}}{2}\right)+\left(\frac{\hat{A}\times\hat{B}+\hat{B}\times\hat{A}}{2}\right)$$

with the first part being hermitian and the second part being antihermitian.

In a similar fashion we may decompose into hermitian and antihermitian parts the dot product:

AB= (AB+BA)/2+(AB-BA)/2. (2.11)

III. THE EXTENSION OF DIRAC NOTATION

By looking at the form of the decomposition into hermitian and antihermitian parts of (2.10) we deduce that $\hat{A}\times\hat{B}$ is hermitian if and only if $\hat{A}\times\hat{B}=-\hat{B}\times\hat{A}$. Similarly AB is hermitian if and only if [A,B]=0. Now we know that if AB is hermitian then we have the commutation law. We are going to prove that if $\hat{A}\times\hat{B}$ is hermitian, then :

(3.1) $$-\left\langle\hat{B}\psi\,|\times|\,\hat{A}\psi\right\rangle=\left\langle\psi\,|\,\hat{A}\times\hat{B}\,|\,\psi\right\rangle$$

Proof of (3.1):

We are going to prove that $\hat{A}\times\hat{B}=-\hat{B}\times\hat{A}$ by taking its cartesian components. For the x component we will have to prove(the rest follows by symmetry considerations):

A_yB_z-A_zB_y=B_yA_z-B_zA_y

By taking a look at the first part we may recall that we have proved (1.1) and (1.9):

$$\left[-\left\langle\hat{B}\psi\,|\times|\,\hat{A}\psi\right\rangle-\left\langle\psi\,|\,\hat{A}\times\hat{B}\,|\,\psi\right\rangle\right]_x=$$

$$=\left\langle\psi\,|\left(B_zA_y-A_yB_z\right)-\left(B_yA_z-B_yA_z\right)|\,\psi\right\rangle$$

The last terms can be rearranged to give –(B_yA_z-B_zA_y)-(A_yB_z-A_zB_y)=0

This means that in the case $\hat{A}\times\hat{B}$ is hermitian we no longer need to take the real part of the bra-ket in external interchangeability of operators.

IV. ALGEBRA OF Re(bra-ket)

Since vector calculus is obeyed by operators concerning the real part of the bra-ket one might be tempted to think that the usual vector identities can be applied. However according to the author's investigations these must be applied with caution. Particularly I have come to the conclusion that each time we transform an expression every member that is multiplied in the left part of the equation must be hermitian in order for these identities to hold. In other words one part of the identity at least must contain hermitian multipliers. For example since the cross product of two real vectors is again hermitian as is the real vector and so is the momentum operator p, indeed we could make use of the following identity:

(4.1) $$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

Using (4.1) we are able to transform:
(4.2) $\hat{p}\cdot(\vec{A}\times\vec{B})=\vec{A}\cdot(\vec{B}\times\hat{p})$

V. ORTHOGONALITY OF THE BASIC HERMITIAN OPERATORS – PROPERTIES OF THE NULL OPERATORS

As a counterexample one could mention that the transformation does not work for the following expression which is part of the square of the Runge-Lenz vector:
(5.1) $\left(\hat{L}\times\hat{p}\right)\cdot(\hat{p}\times\hat{L})$
This is so because the two parts multiplied are not hermitian although L,p are hermitian operators. We may not apply a vector identity in that case although one might argue that in expanding the expression we would find hermitian operators. However one also gets the operator $(pL)^2$ which is double the null operator in our algebra which is problematic for the following reason: Although Re(|pL|ψ>)=0 it still is Im(|pL|ψ>)≠0 and since $(pL)^2$ is hermitian it splits and one (pL) goes to the bra and Re(<ψ|pL|)=0 while Im(<ψ|pL|)≠0 .

(|pL|ψ>)= Re(|pL|ψ>)+i Im(|pL|ψ>)= +i Im(|pL|ψ>) (5.2)

(<ψ|pL)= Re(|pL|ψ>)-i Im(|pL|ψ>) = -i Im(|pL|ψ>) (5.3)

<ψ|pL|pL|ψ>= Re<ψ|pL|pL|ψ>= $\int \mathrm{Im}\left(pL\,|\,\psi>\right)^2 dV$ (5.4)

It is also apparent that :
(5.5) $\mathrm{Re}\left\langle \hat{r}\cdot\hat{L}\right\rangle=\mathrm{Re}\left\langle \psi\,|-i\hbar\vec{r}\cdot\left(\vec{r}\times\nabla\psi\right)\right\rangle=0$

Finally we will find that:

Re<rp>=0. (5.6)

Proof: Its components follow the same rule as its x component:
$2\mathrm{Re}(xp_x)=xp_x+p_xx=[xp_x,-p_xx]=x[p_x,-p_xx]+[x,-p_xx]p_x=$
$=xp_x[p_x,-x]+[x,-p_x]xp_x=0$

VI. REGARDING THE IMAGINARY PART OF NULL OPERATORS

The wavefunction as every complex function may be written as:

$$\psi=|\psi|e^{i\varphi}$$

Actually $\psi=\alpha+i\beta=|\psi|e^{i\phi}=|\psi|\cos\varphi+i|\psi|\sin\varphi$

The Imaginary part of (|pL|ψ>) exists due to the fact that the phase ϕ of the wave function is not single-valued and if we were to calculate |pL|ψ> we would find:

$pl\,|\psi>=i\,|\psi|\nabla\cdot(\vec{r}\times\nabla\phi)=i\,|\psi|\,\vec{r}\cdot\nabla\times\nabla\phi\neq 0$

If we go round a curve f may change by multiples of 2π. Thus if A is the integral over an open surface:

$\vec{A}=\oiint\nabla\times\nabla\phi\cdot d\vec{S}=\oint\nabla\phi d\vec{l}=\delta\phi=2\pi n$, n integer

However the deviations must be zero if φ is to change in a logical manner:

$\iiint(\nabla\cdot\nabla\times\nabla\phi)dV=0$

In the previous expression B is an integral over a closed surface. We conclude that:

$(\nabla\cdot\nabla\times\nabla\phi)$

VII. THE RUNGE-LENZ VECTOR

The work of the author was based on trying to find an alternative way for calculating difficult operators such as is the Runge-Lenz vector. For the sake of being brief we will omit the units:

$$\text{(7.1)}\quad \begin{aligned}\langle K^2\rangle=\mathrm{Re}\langle K^2\rangle&=\left\langle\left(\frac{\hat{L}\times\hat{p}-\hat{p}\times\hat{L}}{2}+\frac{\vec{r}}{r}\right)^2\right\rangle=\\ &=\mathrm{Re}\left\langle\left(\frac{\hat{L}\times\hat{p}-\hat{p}\times\hat{L}}{2}+\frac{\vec{r}}{r}\right)^2\right\rangle=\mathrm{Re}\left\langle\left(\hat{L}\times\hat{p}+\frac{\vec{r}}{r}\right)^2\right\rangle\end{aligned}$$

One may decompose the operator and estimate only hermitian parts for the real part. For our purposes we will make use of the following well known vector identity:

$$\text{(7.2)}\quad (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

By expanding the hermitian form of (7.1) one expects indeed to find hermitian operators:

$$\text{(7.3)}\quad K^2=\left(\frac{\hat{L}\times\hat{p}-\hat{p}\times\hat{L}}{2}+\frac{\vec{r}}{r}\right)^2$$

$$\text{(7.4)}\quad K^2=\frac{2(L^2p^2+p^2L^2)}{4}-\frac{2(Lp+pL)^2}{4}+2\frac{\vec{r}}{r}\cdot\left(\frac{\hat{L}\times\hat{p}-\hat{p}\times\hat{L}}{2}\right)+1$$

The first , third and fourth part of (7.4) by using usual vector identities when taking the real part are found to be exactly what reference books give as:

$$\text{(7.5)}\quad \langle 2L^2(p^2/2-1/r)+1+\hat{X}\rangle=\langle 2\hat{L}^2\hat{H}+1+\hat{X}\rangle$$

There is also one unknown X term missing from (7.5). This is given by the “problematic” second term of (7.4). It is doubly zero so we will use a trick:

$$\text{(7.6)}\quad \mathrm{Re}\langle\hat{X}\rangle=\mathrm{Re}\left\langle-\frac{(Lp+pL)^2}{4}+\frac{(Lp-pL)^2}{4}\right\rangle$$

The first part in the square of the second member of (7.6) has zero imaginary part since it is of the form $(AB+BA)^2$ with zero imaginary part.

Terms in (7.6) are all equal to $<(Lp)^2>=<pL^2>=<pL\psi|Lp\psi>=\int[\mathrm{Im}(Lp\,|\psi>)]^2dV$

By making use of commutation relationships such as $[L_x,p_y]=i\hbar p_z$ one finds:

(7.7) $-[pL,Lp]=[Lp,pL]=i\hbar\hat{p}\cdot\left(\hat{L}\times\hat{p}\right)$

Thus far $\mathrm{Re}\left\langle \hat{X}\right\rangle=\mathrm{Re}\left\langle \dfrac{2i\hbar\hat{p}\cdot\left(\hat{L}\times\hat{p}\right)}{4}\right\rangle$ (7.8)

In reference books one may also find the equality:

(7.9) $\hat{L}\times\hat{p}-\hat{p}\times\hat{L}=2i\hbar\hat{p}$

Using (7.9) we find:

(7.10) $\mathrm{Re}\left\langle \hat{X}\right\rangle=\left\langle -2\hat{p}^2/2-i\hbar\hat{p}\cdot\left(\hat{p}\times\hat{L}\right)/2\right\rangle=\mathrm{Re}\left\langle 2\hat{H}\right\rangle$

The first part of the right member of (7.10) is the familiar <2H> term found in standard textbooks for the electron in the field of a hydrogen like atom since from Virial theorem it is minus two times the mean kinetic energy. The second part is zero because only in the particular problem there is some symmetry concerned. We will estimate it in the following paragraph.

VIII . SPECULATIONS ABOUT HIDDEN VARIABLES

From what was said in paragraph VII and by taking in mind that the deviation of the square of the Lenz vector was quite general one expects that the operator X/2 in (7.8) should be the Hamiltonian at least for central potentials. Next we are going to estimate it:

(8.1) $\left\langle H\right\rangle=\left\langle \dfrac{i\hbar\hat{p}\cdot\left(\hat{p}\times\hat{L}\right)}{4}\right\rangle=1/4\int\psi^*\nabla\cdot[\nabla\times(\vec{r}\times\nabla\psi)]dV$

(8.2) $\left\langle H\right\rangle=1/4\int_0^\infty\nabla\cdot\{\psi^*[\nabla\times(\vec{r}\times\nabla\psi)]\}-\nabla\psi^*\cdot[\nabla\times(\vec{r}\times\nabla\psi)]dV$

We assume that the first part of (8.2) inside the div operator behaves well at infinity and vanishes. Then:

(8.3) $\left\langle H\right\rangle=1/4\int_0^\infty-\nabla\cdot\left(\nabla\psi^*\times(\vec{r}\times\nabla\psi)\right)+(\vec{r}\times\nabla\psi)\cdot\nabla\times\nabla\psi^*dV$

(8.4) $\begin{aligned}&\left\langle H\right\rangle=1/2\int 1/2\left(|\psi|\vec{r}\times\nabla\phi\right)\cdot|\psi|\nabla\times\nabla\phi dV=\\&=1/4\int\left(\vec{r}\times\vec{J}\right)\cdot\nabla\times\nabla\phi\end{aligned}$

The substitutions in (8.4) it is well established that one may associate a true current density J with the probability current density and the magnetization M is one half the position vector cross product the current density J(which is –e/N the probability current density). One might associate then a magnetic field caused by the current density of the electron for then (8.4) would read as minus the magnetic energy. So we would have, where C is constant (accounting for units we have omitted):

$\vec{B}=C\nabla\times\nabla\phi$ (8.5)

(8.6) $\nabla\cdot\vec{B}=0$

Also we know that the time dependence of ψ is associated with constants, the Bohr frequencies therefore has no space dependence.

$$(8.7)\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$$

$$(8.8)\ \vec{B} = \nabla \times \vec{A}$$

$$(8.9)\ \vec{J} = \hbar / m |\psi|^2 \vec{A}$$

In this line of thinking we should assume a polarization current to exist.
The last equation (8.9) is used in superconductors. Another fact is that in that case we would have quantized flux through any surface just like is the case with superconductors.
On the other hand it is known that the polarization is:

$$\vec{P} = -|\psi|^2 \vec{r} \quad (8.12)$$

According to the virial theorem:

$$(8.13)\ \langle -\vec{r}\cdot\nabla U \rangle = \int |\psi|^2 \vec{r}\cdot\vec{F} = -2\langle K.E. \rangle$$

Since the potential is usually electrostatic (even when accounting for interacting dipoles):

$$(8.14)\ \frac{1}{2}\int \vec{P}\cdot\vec{E} dV = < K.E. >$$

The half of the potential energy can be written as the familiar electrostatic energy:

$$(8.15)\ 1/2 < U > = \frac{1}{2}\int |\psi|^2 U dV = -1/2 \int q dV$$

It seems that we should add a +3/2 factor to have the mean potential energy. We will discuss this matter in paragraph X.
Therefore the dielectric susceptibility should be
The polarization is assumed to vary as:

$$(8.16)\ \frac{\partial \vec{P}}{\partial t} = -\vec{r}\frac{\partial |\psi|^2}{\partial t}$$

Finally since the magnetic work involving B is zero and does not influence things and since the magnetic flux of B is quantized B should depend on energy E so that each time a quantum of light is emitted the flux should change accordingly and that is because of the classical law of Faraday that the change of flux is associated with the electric work done.

IX. CONCLUSION- THOUGHTS FOR FURTHER INVESTIGATIONS

Since all these results have been obtained based on the thinking of independency of the space part of the phase φ and |ψ| inside the real part of the volume integral we should check the validity of this line of thought. By estimating the real part of the kinetic energy (which is integrated over the volume) one finds:

$$(9.1)\ \frac{1}{2}\int [\Delta |\psi|^2 + |\psi|^2 (\nabla \phi)^2] dV = \int |\psi|^2 [E - U(r)] dV$$

It sounds familiar from Fourier analysis that half the gradient of phi squared is the kinetic energy. According to our convention:

(9.2) $<K.E.>=-\frac{1}{2}\int \vec{J}\cdot\vec{A}dV$

In this form the magnetic work does not vanish.

Another consequence of this independence is that the magnetic induction B should depend only on φ. So we should write:

(9.3) $\frac{|\psi|^2}{N}\vec{B}=(\vec{M}+\vec{H})$

It should also be true that:

(9.4) $\vec{D}=|\psi|^2\,\vec{E}_{total}=\vec{P}+\varepsilon_0\vec{E}$

(9.5) $|\psi|^2=\frac{\mu_0'}{\mu_0}=\frac{\varepsilon_0'}{\varepsilon_0}$

This way the speed of light remains constant.

APPENDIX A: Proof of an identity used in II

We are going to prove that

$$\vec{A}^*\times\vec{B}-\vec{B}^*\times\vec{A}=2\,\mathrm{Re}\left[\vec{A}^*\times\vec{B}\right] \qquad \text{(A.1)}$$

For let : $\vec{A}=\vec{\alpha}+i\vec{\beta},\mathrm{\vec{B}}=\vec{\gamma}+i\vec{\delta}$

Then we can find for the external products used in (A.1) that:

$$\vec{A}^*\times\vec{B}=(\vec{a}-i\vec{\beta})\times\left(\vec{\gamma}+i\vec{\delta}\right)=\vec{\alpha}\times\vec{\gamma}+\vec{\beta}\times\vec{\delta}+i\left(-\vec{\beta}\times\vec{\gamma}+\vec{\alpha}\times\vec{\delta}\right) \qquad \text{(A.2)}$$

$$\begin{aligned}-\vec{B}^*\times\vec{A}&=-(\vec{\gamma}-i\vec{\delta})\times(\vec{a}+i\vec{\beta})=-\left[\vec{\gamma}\times\vec{\alpha}+\vec{\delta}\times\vec{\beta}+i(\vec{\gamma}\times\vec{\beta}-\vec{\delta}\times\vec{\alpha})\right]=\\&=\vec{\alpha}\times\vec{\gamma}+\vec{\beta}\times\vec{\delta}+i(\vec{\delta}\times\vec{\alpha}-\vec{\gamma}\times\vec{\beta})\end{aligned} \qquad \text{(A.3)}$$

By adding up (A.2) and (A.3) we have the proof of (A.1)