

## The Dirac Equation Without Zitterbewegung

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### Abstract

The prediction of spin is a necessary outcome of relativistic quantum mechanics(RQM). But one must realize that the spin of the electron is contained in non-relativistic physics

as well. In it, one had  $E = \frac{P^2}{2m}$  for a free electron, while the relativistic equation

$E^2 = c^2 P^2 + m^2 c^4$  is required in relativistic physics. We show that the same results may be derived using a non-relativistic treatment as in Dirac's theory but without the usual contradictions, i.e. (zitter bewegung), between the framework of Dirac's theory and SRT.

Key words: electron spin, non-relativistic physics

### 1-Introduction:

A prediction for an important property of the electron, its spin, was contained in Dirac's equation.[1,2]. Despite the success of the equation, there still remained a number of misunderstandings. Fore instance, the spin prediction of the Dirac equation can not be identified without methods of approximation ( Pauli , Fold- Wouthysen)[3]. A short time after its publication, a serious problem was discovered by Schrödinger [4]. The expected velocity of the electron when calculated in the Dirac equation was greater than light speed. This problem was known as (zitter bewegung).

The incorporation of the special theory of relativity(SRT) into quantum mechanics resulted in serious contradictions which remained unsolved until later years. There had been few attempts to eliminate them over the decades. The most prominent attempt was that of Ezzat.G. Bakhoun[5], which required a modification in the mass-energy principle. He proposed a new total relativistic energy formula  $E = mv^2$  instead of Einstein's  $E = mc^2$ , where  $m$  is the relativistic mass of the particle, and  $v$  is the velocity.

Starting from SRT he used the formula,  $E = mv^2$  as total energy to obtain a new Hamiltonian and a new relativistic wave equation[5]. Unlike Dirac's result for zitter bewegung, his result is in agreement with experimental observation and gives the same result for the spin magnetic moment coefficient. Recently, we emphasized the formula

$E = mv^2$  in [6]. In [7] we derived the formula without using the SRT. In this paper, we obtain the same mentioned results as Bakhom starting from the Lorentz force law and relativity principle [6,7,8]. Therefore the spin of electron can be derived automatically in non-relativistic physics without zitterbewegung as is shown in the present paper.

## 2-Derivation the modified Dirac equation from Lorentz Force and Relativity Principle:

As demonstrated also in my paper [6], the relativistic transformation relations are produced without LT and its kinematical effect.

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad , \quad \frac{d\mathcal{E}}{dt} = q\mathbf{E}\mathbf{v} \quad (1)$$

Following the same approach which is used in [6], we can obtain the relativistic transformation equation for momentum, energy, velocity as well as the relativistic factor :

$$P'_x = \gamma(P_x - \frac{u}{c^2}\mathcal{E}) \quad (2a) \quad \mathcal{E}' = \gamma(\mathcal{E} - uP_x) \quad (2b) \quad , \quad P'_y = P_y \quad (2c) \quad P'_z = P_z \quad (2d) \quad \text{Where}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

As well as

$$m = \frac{m_0}{\sqrt{1 - \frac{V^2}{c^2}}} \quad , \quad \mathcal{E} = mc^2 \quad (3)$$

And

$$m' = \frac{m_0}{\sqrt{1 - \frac{V'^2}{c^2}}} \quad , \quad \mathcal{E}' = m'c^2$$

It is simple to prove, that Eqs. (2) and (3) lead to

$$\mathcal{E}'^2 - c^2\mathbf{P}'^2 = \mathcal{E}^2 - c^2\mathbf{P}^2 = m_0^2c^4 \quad (4)$$

we begin deriving the modified Hamiltonian from Eq.(4) i.e.

$$H^2 = c^2\mathbf{P}^2 + m_0^2c^4 \quad , \quad H = \mathcal{E} \quad (5)$$

If we replace  $H$  by  $mc^2$  and  $\mathbf{P}$  by  $mv$  we have

$$m^2c^4 = c^2m^2v^2 + m_0^2c^4$$

If we multiply this expression by  $\frac{v^2}{c^2}$ , we get

$$m^2 v^2 c^2 = m^2 v^4 + m_0^2 v^2 c^2$$

If we now let  $H = mv^2$  we finally have

$$H^2 = c^2 p^2 - m_0^2 v^2 c^2 \quad (6)$$

E.G. Bakhoun started from Eq.(6) and using SRT and the formula  $H = mv^2$  as a total Hamiltonian, obtained a new relativistic wave equation[5]. We will also continue with our method and multiplying, at the same time dividing the last term in Eq.(6) by  $\gamma^2$ , we obtain:

$$H^2 = c^2 p^2 - \frac{c^2 p^2}{\gamma^2} = c^2 p^2 \left(1 - \frac{1}{\gamma^2}\right)$$

where  $H = mv^2$ , and  $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ .

But  $1 - \frac{1}{\gamma^2} = \frac{v^2}{c^2}$ , so we get finally.

$$H^2 = p^2 v^2 \quad (7)$$

Following the same method as Dirac, we can write the Hamiltonian  $H$  as a linear combination of momentum components.

$$\hat{H} = v \hat{\alpha}_\mu p_\mu \quad (8)$$

The matrices  $\alpha_\mu$  must satisfy the condition  $\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 2\delta_{\mu\nu}$

So  $\alpha_\mu$  are the same as Dirac matrices.

### 3- Derivation of Pauli Equation without the approximations methods

Now we find the eigenvalues and eigenfunctions of Eq.(2), which can be written as follows:

$$i\hbar \frac{\partial}{\partial t} \psi = v \alpha_j p_j \psi \quad (9)$$

The solutions of Eq.(9) are plain waves which have the following form.

$$\psi(x, t) = \psi(x) e^{-\frac{i}{\hbar} \epsilon t} \quad (10)$$

Substituting Eq.(4) in Eq.(3), and considering

$$\psi(x) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \quad (11)$$

where  $\varphi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$  and  $\chi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$  are two-component spinors.

we find:

$$\varepsilon \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = vp_j \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \Rightarrow \begin{cases} \varepsilon \varphi = vp_j \sigma_j \chi \\ \varepsilon \chi = vp_j \sigma_j \varphi \end{cases} \quad (12)$$

We look again for a plane-wave solution of the form

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} e^{\frac{i}{\hbar} p x} \quad (13)$$

Substituting Eq.(13) in Eq.(12), we obtain:

$$\begin{cases} \varepsilon \varphi_0 - vp_j \sigma_j \chi_0 = 0 \\ \varepsilon \chi_0 - vp_j \sigma_j \varphi_0 = 0 \end{cases} \quad (14)$$

These two equations have a solution if:

$$\begin{vmatrix} \varepsilon & -vp_j \sigma_j \\ -vp_j \sigma_j & \varepsilon \end{vmatrix} = 0 \quad \text{i.e.} \quad \varepsilon^2 = v^2 p^2 \quad \text{or,} \quad \varepsilon = \mp |vp| \quad (15)$$

From Eq.(14) we find:

$$\chi_0 = \frac{vp_j \sigma_j}{\varepsilon} \varphi_0 \quad \text{and} \quad \varphi_0 = \frac{vp_j \sigma_j}{\varepsilon} \chi_0$$

From these two equations we get:

$$\chi_0 = \frac{(vp_j \sigma_j)^2}{\varepsilon^2} \chi_0$$

Using Eq.(15) in the last equation, we get:

$$\chi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In a same way we can obtain:

$$\varphi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \varphi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From the above calculations we find:

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = N \begin{pmatrix} \frac{vp_j \sigma_j}{\varepsilon} \chi_0 \\ \chi_0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = N \begin{pmatrix} \varphi_0 \\ \frac{vp_j \sigma_j}{\varepsilon} \varphi_0 \end{pmatrix} \quad (16)$$

By calculating the normalization constant N For positive and negative energy solutions, we find

$$N = \sqrt{\frac{R \pm vp}{2R}}$$

Where  $\varepsilon = \pm R$  and  $R = |vp|$ :

If we follow the usual procedure to introduce a magnetic field into the Dirac equation for a free electron :

$$\vec{p} \rightarrow \vec{p} - \frac{e}{c} \vec{A} \quad , \quad \vec{B} = \vec{\nabla} \wedge \vec{A} \quad (17)$$

So we are concerned with solving Eq.(9) in the following form:

$$H\psi = v \alpha \left( \vec{p} - \frac{e}{c} \vec{A} \right) \psi$$

Considering Eq.(6), we get.

$$H\varphi = v \vec{\sigma} \left( \vec{p} - \frac{e}{c} \vec{A} \right) \chi \quad (18)$$

$$H\chi = v \vec{\sigma} \left( \vec{p} - \frac{e}{c} \vec{A} \right) \varphi \quad (19)$$

From Eq.(19) we find:

$$\chi = \frac{v}{H} \vec{\sigma} \left( \vec{p} - \frac{e}{c} \vec{A} \right) \varphi \quad (20)$$

Substituting Eq.(20) in Eq.(18), we obtain:

$$H\varphi = \frac{1}{m} \left( \vec{\sigma} \left( \vec{p} - \frac{e}{c} \vec{A} \right) \right)^2 \varphi \quad (21)$$

In Dirac's treatment of the subject, he was able to show that the following equation holds

$$\left( \vec{\sigma} \left( \vec{p} - \frac{e}{c} \vec{A} \right) \right)^2 = \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 - \frac{e\hbar}{c} \vec{\sigma} \cdot \vec{B}$$

Substituting the last equation in the Eq.(21), we obtain:

$$H\varphi = \frac{1}{m} \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 \varphi - \frac{e\hbar}{mc} \vec{\sigma} \cdot \vec{B} \varphi \quad (22)$$

The Dirac equation for an electron in an external magnetic field is well known to yield the Pauli equation in the low-energy limit [ ]. So the second term on the r.h.s. of Eq.(22) is the term that represents the interaction of the field with the electron's magnetic moment. This prediction of electron spin in the non-relativistic limit provides the property of the Dirac equation only. That is not true if one can derive the spin from classical quantum approach as it is shown in this paper. An important result for Eq.(22) is also that we did not use any kind of approximations to reach it, while this is not the case with Dirac's equation and the Bakhom's approach. Therefore we have gotten the mass increase  $m$  but not the rest mass  $m_0$  in Eq.(22).

#### 4-Calculating the velocity

Another conceptual misunderstandings about Dirac's equation was the zitterbewegung and the result that a velocity measurement of a Dirac particle at any instant in time is  $\pm c$ . This result is of course in disagreement with experimental observation so the velocity will be  $\pm v$ .

Starting from Eq.(9):

$$i\hbar \frac{\partial}{\partial t} \psi = -i\hbar v \alpha_j \nabla_j \psi \quad (23)$$

Multiplying Eq.(23) from left with  $\psi^+$  give:

$$i\hbar \psi^+ \frac{\partial}{\partial t} \psi = -i\hbar v \psi^+ \alpha_j \nabla_j \psi \quad (24)$$

By taking the hermetic conjugate of Eq.(23), we have

$$-i\hbar \frac{\partial}{\partial t} \psi^+ = i\hbar v \nabla_j \psi^+ \alpha_j \quad (25)$$

Multiplying Eq.(25) with  $\psi$  from right gives:

$$-i\hbar \frac{\partial}{\partial t} \psi^+ \psi = i\hbar v \nabla_j \psi^+ \alpha_j \psi \quad (26)$$

Subtracting both equations (24) and (26), we get

$$\frac{\partial}{\partial t} (\psi^+ \psi) + \nabla_j (\psi^+ v \alpha_j \psi) = 0 \quad (27)$$

This is the known form of continuity equation  $\frac{\partial}{\partial t} \rho + \nabla \cdot \vec{J} = 0$ , Where  $\vec{J} = v \psi^+ \vec{\alpha} \psi$  is the current density, and  $\rho = \psi^+ \psi$  is the particle density.

But it is known that  $\vec{J} = \rho \vec{v}$ , hence we deduce that the velocity of the particle wave is the same as the velocity of the particle itself, since the eigenvalues of the matrices  $\alpha_j$  are  $\pm 1$ .

### Summary

Unlike Dirac's equation and its predications, our modified Dirac equation i.e. Eq.(9) and its results removes the conceptual difficulty of a non-relativistic limit and zitterbewegung to derive the spin without using any kind of approximations. We derive the velocity of the particle wave and show it is the same as the velocity of the particle itself, but not  $\pm c$ .

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