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Archimedes Principle in Completely Submerged Balloons: Revisited

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Abstract

It is observed in critical analysis of completely submerged floating balloons in water, that under some feasible conditions, the volume of fluid filled in balloon takes indeterminate form i.e. V =0/0 in equations based upon Archimedes principle. These equations became feasible 1935 years after enunciation of the principle in 1685, when Newton defined g in *The Principia.* If in this case definition of the principle is generalized i.e. upthrust is proportional to the weight of fluid displaced, then results are consistent i.e. V=V. Thus co-efficient of proportionality, f comes in picture, which accounts for factors not taken in account by the principle e.g. for shape of body, viscosity of medium, magnitude of medium, surface tension etc. Stokes law is justified under some assumptions and takes in account the shape of body and viscosity of medium. The value of f depends upon such factors. Furthermore some specific experiments have been suggested to confirm effect of coefficient of proportionality. Such specific studies do not mean any comment or conclusion of the established status of the principle.

1.0 Completely submerged floating balloons lead to indeterminate volume of fluid filled in them

Archimedes principle was stated in 250BC, and serves a method for determination of densities of bodies [1]. It is also used for determining the conditions of floatation. But under certain conditions which can be easily experimentally achieved, the volume of medium filled in balloon becomes indeterminate i.e. V=0/0. It is not justified. The exact volume V is obtained if the principle is generalized. The generalized form of the principle is

'the upthrust is proportional to the weight of fluid displaced by body'

$$U = fVD_m g \qquad (1)$$

where V is volume of body, D_m is density of medium displaced , g is acceleration due to gravity. f is co-efficient of proportionality which depends upon the factors which are not taken in account by the principle i.e. shape of floating body, coefficient of viscosity of medium and surface tension etc. Experiments and mathematical equations have been mentioned to justify the inception of f. If f =1 then eq.(1) becomes original form of Archimedes principle i.e $U = VD_m g$

Resultant weight (weight of body in fluid) = weight – upthrust = $(D_b - D_m)Vg$ (2)

The body floats if the resultant weight is zero i.e.

$(D_b - D_m)Vg$ or $D_b = D_m$ (3)

If $D_b > D_m$ body sinks or falls in medium and if $D_m < D_b$, body rises upward in medium [2-3]. The modern mathematical equations became feasible on 2265 years old Archimedes principle, after 1935 years of its enunciation [4-5] when Newton [6] published *The Principia* and defined acceleration due to gravity, g in 1685. Now when these are critically analysed then inconsistent results are obtained and principle is purposely generalized.

Floating balloon: Consider a balloon filled with medium of density D_m floating in water of density D_w. The volume of sheath of balloon /vessel is v which also includes volume of mass if any attached to it. The mass corresponding to volume v is m. The volume of medium filled inside the balloon is V (say air, wood, metal and gases). According to Archimedes principle the upthrust experienced by balloon is equal to weight of fluid displaced [2-5]. The body displaces fluid equal to own volume. The condition of floatation of balloon is that resultant weight becomes zero i.e.

$V D_m g + mg = (V+v) D_w g$ (4)

Or $m = (V+v) D_w - V D_m$ (4)

Borowitz and Beiser has quoted similar equation (general) but neglecting volume v which is not quantitatively justified [3]. Thus,

$M = V D_w - V D_m$

or $V = \frac{M}{(D_w - D_m)}$ (5)

1.1 Indeterminate form of volume

(i) It is basic requirement of equation (equality of LHS and RHS) having n parameters, that if (n-1) parameters are given then, the remaining parameter (n_{th} say) can be determined. It is basic principle of algebra. If it is not so, then equation is inconsistent. For example in law of gravitation, $F = Gm_1m_2/r^2$, if all other parameters except m_1 are given, then value of m_1 can be calculated i.e.

$m_1 = Fr^2/Gm_2$ (6)

Now volume V is required to be calculated which can support mass equal vD_m in water. From eq.(4) the value of volume can be written as

$$V = \frac{(m - vD_w)}{(D_w - D_m)} \qquad (7)$$

Like wise equations D_m, D_b and v can be written. If the m, D_m, D_w and v are given in equation then V must be calculated from eq.(7) under experimentally consistent and achievable conditions. If it is not so then it is limitation of Archimedes Principle, not that of interpretation.

(ii) Now we can try to calculate the volume (V) of fluid filled in balloon (air, wood, metal and gases, say); the sheath of balloon is having volume v and mass m (vD_w) , such that density of fluid filled inside is equal to that of water ($D_m = D_w$). As in case of law of gravitation, in this case also experimentally consistent result is expected. Obviously volume should turn out equal to V, which is actual value. Hence substituting, mass from eq.(4) i.e. $m = vD_w$ in eq.(7), we get

$$V = \frac{(vD_w - vD_w)}{(D_w - D_w)} = \frac{0}{0} \qquad (8)$$

which is indeterminate form i.e. volume of medium filled in balloon (wood, metal, air and gases) becomes undefined but in actual experimental set up volume is V consisting of metal, wood and gases. Thus there is singularity in equation of volume V under this condition, hence other variables (D_m, D_w, and v etc.) cannot be determined.

(a) If V is in indeterminate form, then there is singularity in equations of Archimedes principle, under this specific condition. Thus RHS of eq.(8) becomes devoid of units and dimensions which is not defined. The dimensions of the LHS are $M^0L^3T^0$ and units m^3. Incidentally the dimensions of LHS and RHS of eq.(4) are of mass i.e. ML^0T^0 and units kg. But the eq.(8) which is derived from it is dimensionally inconsistent, hence eq.(4) must be re-defined under this condition. The dimensional homogeneity is the first and foremost condition for consistency of an equation and all equations have to obey it without exception. If any equation does not obey it then that equation loses its identity.

(b) Every equation needs to be checked for its consistency for various values of involved parameters. Although division by zero is not permitted, yet it smoothly follows from equations based upon Archimedes' principle as per its definition. Also in this case numerator of the equation also becomes zero. So it is only and only limitation of Archimedes principle. The conditions of interpretation can be experimentally achieved in number of ways, so interpretation is not under hypothetical conditions.

In case due to certain reasons the equation does not give consistent results, then it does not mean it must not be interpreted under that condition. An equation has to inherently obey many conditions. This discussed situation i.e. equality of density of medium filled in balloon and that of density of water ($D_m = D_w$) can be achieved in many cases. Experimentally the volume of medium (say air, wood, metal and gases) filled in balloon is V not 0/0. This situation can be obtained in number of ways. This equation in this particular case is not applicable. This intrigue can only be solved by generalizing Archimedes principle. Thus this aspect regarding Archimedes principle remained unstudied, hence interpreted here. Such specific studies do not mean any comment or conclusion on the established status of the principle in other cases.

The mathematical equations (hence definition of the principle) should be such that this situation should not arise, and mathematically exact volume V is obtained. Further v, D_m, D_w cannot be calculated as volume is indeterminate i.e. V = 0/0 as eq.(8) attains singularity. This experiment or perception has not been discussed even by Batchelor [7] in standard treatise*, An Introduction to Fluid Dynamics* in chapter I in relevant section 1.4, A body 'floating' in fluid at rest. This indicates the gravity and originality of the discussion. Einstein introduced a term cosmological term in his cosmological thesis, which became zero under some conditions.

Einstein while writing his thesis on static universe divided with cosmological term which became zero under some conditions. It was pointed out by Friedmann. Later on Einstein abandoned the concept calling it as the biggest blunder, according to George Gammow [8]. Thus alternate concept of non-static universe was accepted. Thus Einstein did not insist that his thesis involves division by zero is correct, and Friedman who showed the proof involves division by zero is wrong. It is obvious that if an equation involves division by zero, then it is limitation of equation, not that of interpreter.

2.0 Generalization of Archimedes Principle

It is confirmed in eq.(8) that under some achievable conditions, the volume becomes indeterminate which is not justified. Under above conditions (in application to floating bodies) Archimedes principle becomes invalid mathematically. Thus the principle has to be validated for all conditions; it can be so if definition of Archimedes principle is generalized. So the alternative follows from the principle itself. The generalized form of the principle [4] is

' upthrust experienced by body is proportional to the weight of fluid displaced'

$$U_{gen} \propto (V+v) D_w g \quad \text{or} \quad U_{gen} = f(V+v) D_w g \qquad (9)$$

where f is co-efficient of proportionality. The author [4-5] has already generalized the principle in view of the results obtained from the first stage experiments involving air filled balloons of different shapes floating in water. In such experiments the mass which balloon supported is found to depend upon the shape of balloon (spherical, long pipe, and umbrella shaped etc.). In preliminary or first stage experiments, an umbrella shaped balloon supported more mass (in general sense weight) than long pipe shaped balloon [4]. However, it is clearly added that final conclusion must be drawn from the specific, repeatable sensitive experiments , as preliminary experiments simply give qualitative trends. The subtle results from preliminary experiments compels some extremely sensitive experiments with sophisticated instruments[4], as under some conditions the same equation gives indeterminate form of volume i.e. Eq.(8). For final confirmation such experiments require the most sensitive and sophisticated equipments.

Coefficient of Proportionality f : f is co-efficient of proportionality like numerous others in science. If f=1 then both the generalized and original forms are the same. Its value is experimentally measured and depends upon the inherent characteristics of the experimental variables. The physical significance of the coefficient of proportionality can be understood by both methods.

Theoretically using the generalized form of Archimedes principle the exact volume V is obtained as in eq.(13). Thus the various other values of D_m, D_b, v etc can be calculated. Hence singularity is removed in the equations under the given conditions. Hence the value of f is theoretically justified. The magnitude of f can be determined experimentally in various experiments. Now in this specific case the condition for floatation using generalized upthrust can be written as

$$V D_m g + mg = f (V+v) D_w g \quad (10)$$

Now equations for mass m, and volume V can be written as

$$m= f (V+v) D_w - VD_m \quad (11)$$

$$V = \frac{(m - fvD_w)}{(fD_w - D_m)} \quad (12)$$

Under the similar condition ($D_m = D_w$), the exact volume is obtained.

$$V = \frac{(f(V+v)D_w - VD_w - fvD_w)}{(fD_w - D_m)} = \frac{(f-1)D_w V}{(f-1)D_w} = V \quad (13)$$

Now consistent results are obtained if value of f is different from unity. Also now correct values of v, D_m and D_w are obtained. Unlike eq.(8), in this case, division by zero is not involved. Also numerator is non-zero, hence consistent and logical result is obtained. The condition of floatation is the same i.e. $D_m = D_w$. The dimensions and units are same in both LHS and RHS i.e. $M^0L^3L^0$ and m^3. Then these equations based upon the generalized form of Archimedes principle are mathematically consistent and don't have any limitation like these equations based upon Archimedes principle. Hence mathematically the generalization is justified. The magnitude of f can be determined experimentally.

Experimentally Archimedes principle has many applications, eq.(2) is also used to explain motion of rising, falling and falling bodies qualitatively. The density of body, density of medium, shape of body, magnitude of medium, surface tension etc are the various relevant parameters, when experiments are explained using Archimedes principle. In eq.(3) only D_m and D_b are taken in account; the other factors such as shape of body, viscosity of medium, magnitude of medium, surface tension etc. are taken in account by the coefficient of proportionality f. Thus the generalized form of Archimedes principle is the complete principle in this regard.To confirms such factors specific experiments are required.

3.0 Physical significance of co-efficient of proportionality or its experimental validity.

Archimedes Principle was initially established for measurement of densities of bodies, by attaching body to balance with help of string for weighing. But the rising, falling and floating bodies are not externally attached to balance/instrument. This is conceptual difference between original verification of Archimedes principle and its further applications. The applications of Archimedes principle i.e. eq. (2), are also extended to rising, falling and floating bodies. Hence value of f can be discussed in all such cases. Further original and generalized forms of Archimedes principle only differ by value of f. More the difference between f and unity, more significant will be the generalized form of Archimedes principle.

3.1 **Floating bodies**

According to Archimedes principle i.e. eq.(3) for floating bodies only densities of bodies and media are relevant, rest all others factors e.g. shape of body, magnitude, surface tension , viscosity of medium etc. are irrelevant. In terms of the generalized form of Archimedes principle,

weight of body = upthrust exerted by medium (buoyant force)

$$VD_b g = f\, VD_m g \qquad (14)$$

or $$D_b = f\, D_m \qquad (15)$$

Thus when body floats then f takes in account other factors (other than D_m and D_b) e.g. shape of body, magnitude of medium, surface tension , viscosity of medium etc. Thus theoretically generalized form of Archimedes principle is complete, as it takes all possible involved factors in account and original form only takes in account D_b and D_m. The effect of these factors in such phenomena can be confirmed in specifically designed experiments. It can be confirmed by fabricating bodies of non-hygroscopic nature [8] of different shapes (flat or umbrella shaped or distorted shape); if such bodies float in water of slightly more destiny then value of f other than unity will be confirmed. Then value of f will be D_b / D_m.

Prospective Experiment: The effect of shape may be specifically tested for completely submerged floating balloons/bodies in more viscous fluids. Normally the density of glycerine 1.26 times that of water but the co-efficient of viscosity of glycerine is1058 times that of water. If a body (typically flat, distorted or umbrella shaped) of density 1.26001gm/cc floats (completely submerged) in glycerine of density 1.26gm/cc, then coefficient of proportionality from eq.(15) can be calculated as 1.000008. The systematic study of viscosity started in the beginning of the 19^{th} century [10] much after enunciation of Archimedes principle. Hence effect of viscosity has to be checked in all aspects. Generally effect of viscosity was studied with equation of viscous force (F = 6πηrc, η coefficient of viscosity, r radius of sphere, and c is constant velocity) and Archimedes principle. The co-efficient of viscosity is related with force F and constant velocity c. In addition, the generalized form of Archimedes principle takes effect of viscosity in account explicitly via the coefficient of proportionality. In case of static floating bodies (c =0) the effect of viscous force is irrelevant.

3.2 Stokes' Law and Arnold's experiments confirm significance of effects of shape and viscosity of medium on falling bodies.

This discussion is mainly addressed to completely submerged floating bodies in water. The experiments have been suggested to determine the value of f. In addition the same (effects of shape of body, viscosity of medium etc.) can be understood in case of falling bodies as discussed below.

Falling bodies. Stokes in 1845 put forth that under five postulates [11] small spheres of radius r, in fluid of coefficient of viscosity η fall with constant velocity c or zero acceleration, which is given by

$$c = \frac{2r^2\left(1 - \frac{D_m}{D_b}\right) g D_b}{9\eta} \qquad (16)$$

The eq.(16) is obtained with help of Archimedes principle and viscous force F = 6πηrc (η is co-efficient of viscosity) under certain assumptions. In 1910 Arnold [11] verified eq. (16) in water with an accuracy of a few tenths of 1% for sphere of rose metal of radii 0.002 cm i.e. V= 33.524×10^{-9} cc. Thus it is applicable in extremely narrow range. Thus sphere of radii more than 0.002 cm fall with a variable velocity or motion is accelerated. The small spheres attain constant velocity due to viscous force of fluid. Thus it is concluded that shape of body and viscosity of medium (even magnitude of medium) influence the results of falling bodies e.g. sometimes bodies move with constant velocity (a=0) or variable velocity. it is confirmed experimentally while justifying Stokes Law. These observations are consistent with the generalised form of the principle. Hence such experiments can be conducted over wide range.

Archimedes principle implies that body sinks in medium if resultant weight is positive i.e. density of body is more than that of medium ($D_b > D_m$). It is evident from eq.(2). In general if we drop a flat steel sheet (5m × 5m or distorted shape) of mass 1kg or spherical body of steel of mass 1kg. Then spherical body falls quickly than flat sheet (5m × 5m or distorted shape) of mass 1kg. Both sphere of steel and flat sheet have same weight, upthrust, resultant weight (hence resultant acceleration) should fall equal distances in equal times, according to Archimedes principle. But the flat body or sheet falls slowly than spherical body, it is due to shape of body. Similar results can be obtained if bodies of different masses are studied. The similar explanation may be identically given for rising bodies and results can be critically checked. Thus generalized form of Archimedes principle is useful in understanding or explaining such phenomena.

3.3 Mathematical derivation of Archimedes principle is only for symmetric shaped body.

The effect of shape of body is not only significant in application of the principle but also in its mathematical derivation [2-3]. Consider in fluid of density D_m, block of height H floats (under precisely static conditions and density of fluid is precisely uniform) such that upper surface of body is at depth h in fluid. Also areas of upper and lower surfaces of body are regarded as precisely equal (say, A). The upthrust experienced by block [2-3] is difference between thrusts at lower and upper surfaces i.e.

u =Thrust at lower surface –thrust at upper surface

$$u = \{p+D_m g(h+H)\} A - (p+D_m gh) A = D_m HAg \quad (17)$$

$$\text{or } u = D_m Vg = \text{weight of fluid displaced} \quad (18)$$

Now upthrust is equal to weight of fluid displaced if areas of upper and lower surfaces of the block are the same. But this derivation is not applicable for body of arbitrary shape, hence shape has a role to play even in theoretical derivation of Archimedes principle. The similar conclusions can be drawn from applications of Archimedes principle. Thus some specific and sensitive experiments are required to determine effects of shape of body and viscosity of medium i.e. value of f in the generalized form of Archimedes principle. The various results are shown in Table I regarding original and generalized forms of Archimedes principle.

Table I : Consequences of indeterminate form of volume.

Sr. No.	Characteristic	Original form of Archimedes principle	Generalized form of Archimedes principle
1	Definition	$U=VD_mg$	$U=fVD_mg$
2	Volume under certain conditions	$V=\frac{0}{0}$	V=V (500cc, say)
3	Co-efficient of proportionality	f=1	f< 1 or f >1 or f =1
4	Status of parameters.	Does not account for shape of body and viscosity of medium.	Takes in account the shape of body and viscosity of medium.
5	Specific experiments	Break down under certain conditions	Some specific experiments are suggested to determine effects of shape and viscosity of medium for completely submerged floating bodies. In typical experiment value of f is assessed as1.000008.

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