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R_4 Embedded Into E_5 with Perfect Fluid

J. H. CALTENCO, J. LÓPEZ-BONILLA, R. PEÑA-RIVERO

Sección de Estudios de Posgrado e Investigación
Escuela Superior de Ingeniería Mecánica y Eléctrica
Instituto Politécnico Nacional
Edif. Z, 3er Piso, Col. Lindavista, C.P. 07738 México, D.F.
Email: hcalte@maya.esimez.ipn.mx

ABSTRACT: We consider spacetimes with perfect fluid embedded into E_5, and it is showed that the Newman-Penrose formalism [1-3] gives simple proofs for results obtained by Krishna Rao [4] and Barnes [5].

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1. - INTRODUCTION.

We here shall employ the notation and quantities of [3,5-11].

The curvature of the spacetime is generated by a perfect fluid via the Einstein equations:

$$G_{ab} + \lambda g_{ab} = -(\rho + p) u_a u_b - p g_{ab} \quad , \quad u^r u_r = -1 \quad \textbf{(1)}$$

where λ is the cosmological constant and u_j, ρ, p are the 4-velocity, density and presion of the fluid, respectively. The present work is dedicated to metrics which are solutions of (1) and class one (that is, they accept embedding into E_5 [3,6]). Thus, it is convenient a brief resumé of interesting results in the theme under study, therefore:

a). Szekeres [12] and Greenberg [13] proved that when $\lambda = p = 0$, then (1) not accepts geometries with Petrov type N.

b). Wainwright [14] realized a complete analysis of algebraically special metrics satisfying (1), and such that their degenerate Debever-Penrose vector [3] defines a null congruence geodesic without shear and expansion. In particular, he deduced that:

"The Gödel metric [15] is the unique solution of (1) with Petrov type ≠ *O, I* and

p=0, whose repeated null congruence is geodesic with zero expansion and shear", **(2)**

and it is possible to show that the Gödel metric not admits embedding into E_5 [8,16-18].

c). Wainwright [19] and Carminati-Wainwright [20] investigated solutions of (1) with *p = 0* and Petrov type *D*.

d). The only known metric Petrov type III verifying (1), for $\lambda = 0$, was constructed by Allnut [21] and its corresponding degenerate null congruence is geodesic, has shear and expansion but without rotation.

e). Bonnor-Davidson [22] obtained a solution of (1) with $\lambda = 0$ and Petrov type II, and its repeated principal congruence is geodesic, non-zero expansion but without shear and rotation.

f). Stephani [23] used the embedding process to determine all solutions of (1) with Petrov type *D,* class one and zero acceleration of the matter. Besides, Stephani suggested that any R_4 with perfect fluid and Petrov type *O* has class one, which is correct only when $\rho \neq \lambda$ [6].

g). Barnes [5] employed the Gauss equation to study the compatibility between the Petrov and Churchill-Plebañski [24-28] types for the Weyl, Ricci and second fundamental tensors, thus he showed that:

"All perfect fluids of class one have Petrov types *D* or *O* and b_{ac} must be

[11(11)] or [1(111)], respectively", **(3)**

which implies that the mentioned metrics of Allnut [21] and Bonnor-Davidson [22] do not accept embedding into E_5.

Besides, from (3) it is immediate the following theorem of Pokhariyal [29]:

"Any spacetime of class one whose elementary divisors of

b_{ij} are reals (but not simples), do not admit of a perfect fluid distribution". **(4)**

h). Krishna Rao [4] studied (1) when $\lambda = 0$ and he proved the results:

"All perfect fluids with Petrov type *O* and spherical symmetry,

have class one", **(5)**

and

"All perfect fluids with state equation $\rho = 3p$, and spherical symmetry, have

Petrov type *O* and class one". **(6)**

The interior solution of Schwarzschild can be embedded into E_5 [30] due to (5). Tikekar [31] showed a metric satisfying (6).

i). Szekeres [16] obtained two interesting theorems:

"If R_4 with perfect fluid has class one, then the fluid must be without rotation", **(7)**

and

"If R_4 embedded into E_5 has a perfect fluid with $p = 0$, then this spacetime

is of the Friedman (Petrov type *O*)", **(8)**

both theorems (7) and (8) imply that the Gödel metric [15] (perfect fluid with rotation and Petrov type *D*) has not of class one [8,17,18,30].

Goldman-Rosen [32] have used perfect fluids embedded into E_5 to construct cosmological models in general relativity.

In the next Section we shall employ the Newman-Penrose (NP) formalism [1-3] for the analysis of metrics verifying (1) with class one, resulting thus one relation of Krishna Rao [4] and a part of the theorem (3) of Barnes [5].

2.- Equations of Gauss and Codazzi.

A spacetime accepts isometric and local embedding into E_5 if and only if there is the second fundamental form $b_{ac} = b_{ca}$ satisfying the equations [3,6]:

$R_{ijkc} = \varepsilon \left(b_{ik} b_{jc} - b_{ic} b_{jk} \right)$ Gauss **(9)**

$b_{ij;k} = b_{ik;j}$ Codazzi **(10)**

where $R_{\alpha c r j}$ represents the curvature tensor, $\varepsilon = \pm 1$ and *;r* means the covariant derivative.

Collinson [17,33] used (9) to demonstrate the identity:

$$ {}^{*}R^{jm}{}_{qt}\, R_{jmpa} = -\frac{k_2}{12}\eta_{qtpa} \quad \textbf{(11)} $$

for any R_4 of class one, where ${}^{*}R_{ijrc}$ is the simple dual of Riemann tensor, η_{abcd} is the Levi-Civita tensor and $k_2 = {}^{*}R^{*}{}_{ijrc} R^{ijrc}$ is an invariant of Lanczos [34,35] in terms of the double dual ${}^{*}R^{*}{}_{jmpa}$. The projection of (11) onto a null tetrad of NP leads to a set of 14 equations (which are explicitly in [36]) for the quantities ψ_a and ϕ_{ab} (NP components of the Weyl and Ricci, tensors respectively), and their complex conjugates $\overline{\psi}_a$ y $\overline{\phi}_{ab}$.

We now shall employ the NP formalism to study perfect fluids of class one. Thus, we construct an orthonormal real tetrad with $e_{(4)b} = u_b$, that is, the unitary temporal vector coincides with the 4-velocity of the fluid. We define the NP tetrad in the usual manner ($i = \sqrt{-1}$):

$$ m^r = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^r - i\, e_{(2)}{}^r\right) \quad , \quad \overline{m}^r = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^r + i\, e_{(2)}{}^r\right) \quad , $$
$$ l^r = \frac{1}{\sqrt{2}}\left(e_{(4)}{}^r - e_{(3)}{}^r\right) \quad , \quad n^r = \frac{1}{\sqrt{2}}\left(e_{(4)}{}^r + e_{(3)}{}^r\right) \quad . \qquad \textbf{(12)} $$

If we project (1) onto (12), then:

$$ R = -\rho + 3p + 4\lambda \quad , \quad \phi_{00} = \phi_{22} = 2\phi_{11} = -\frac{1}{4}(\rho + p) \neq 0 \quad \textbf{(13)} $$

and the another ϕ_{ar} are zero; we suppose that $(\rho+p)\neq 0$ because if $(\rho+p)=0$ then R_4 should be an Einstein type spacetime, but it is known [16] that these space must be the DeSitter model.

When we use (13) in the 14 NP equations [36] (which are equivalent to (11)) result the restrictions:

$$\psi_0 = \overline{\psi}_4 \quad , \quad \psi_1 = -\overline{\psi}_3 \quad , \quad \psi_2 = \overline{\psi}_2 \quad ,$$

$$\psi_0\,\overline{\psi}_1 + \psi_1\left(\psi_2 + \frac{R}{6}\right) = 0 \ ,$$

(14)

$$\psi_0\left(-\psi_2 + \frac{R}{12}\right) + \psi_1^2 = 0 \ ,$$

$$3\psi_2\left(\psi_2 + \frac{R}{6}\right) - \psi_0\,\overline{\psi}_0 + 2\psi_1\,\overline{\psi}_1 = 0 \ ,$$

which imply that:

$\psi_1 = \psi_3 = 0$, **(15)**

then the system (14) adopts the form:

$$\psi_0\left(-\psi_2 + \frac{R}{12}\right) = 0 \qquad , \qquad 3\psi_2\left(\psi_2 + \frac{R}{6}\right) - \psi_0\,\overline{\psi}_0 = 0 \quad \textbf{(16)}$$

and thus we have two possibilities:

A). $\psi_0 = 0.$

The equations (14) and (16) lead to $\psi_4 = 0$ with:

$$\psi_2\left(\psi_2 + \frac{R}{6}\right) = 0 \quad , \quad \textbf{(17)}$$

relation obtained by Krishna Rao [4] for the case of spherical symmetry, we here do not need some symmetry. The condition (17) permits the options $\psi_2 = 0$ (Petrov type *O*) and $\psi_2 = -\frac{R}{6} \neq 0$ (Petrov type *D*).

B). $\psi_0 \neq 0.$

Then (14) and (16) imply the relations:

$$\psi_4 = \overline{\psi}_0 \quad , \quad \psi_0 \overline{\psi}_0 = \frac{R^2}{16} \neq 0 \quad , \quad \psi_2 = \frac{R}{12} \neq 0 \quad \textbf{(18)}$$

which jointly with algorithms [3,37-39] to determine the Petrov type give us the type *D*.

The expressions (17) and (18) are the only alternatives for a perfect fluid, thus we have obtained a part of the theorem:

"All perfect fluids (with any cosmological constant) of class one

are type *D* or conformally flat", **(19)**

shown by Barnes [5] via the Churchill-Plebañski algebraical classification [24-28], we here do not need such classification.

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