the general science journal

R_3 (R_4) OF EINSTEIN EMBEDDED INTO R_4 (R_5) OF CONSTANT CURVATURE

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ABSTRACT.

The Einstein spaces R_3 and R_4 are umbilical when they are locally and isometrically embedded (class one) in a riemannian space of constant curvature is proved.

1. INTRODUCTION.

It's deal with riemannian spaces of three and four dimensions of Einstein type, that is, those where the Ricci tensor R_{ac} is proportional to the metric tensor g_{ac}:

$$R_{jk} = \frac{R}{n} g_{jk}, \quad n=3,4. \quad \textbf{(1)}$$

If these spaces are locally and isometrically embedded into another (n+1)-dimensional riemannian space of constant curvature K, then the Gauss-Codazzi equations are verified [1-3]:

$$R_{ijrm} — K(g_{ir}g_{jm} — g_{im}g_{jr}) = \varepsilon\,(b_{ir}b_{jm} — b_{im}b_{jr}), \quad \textbf{(2)}$$

$$b_{ij;r} - b_{ir;j} = 0, \quad \textbf{(3)}$$

where $\varepsilon \pm 1$, R_{acij} is the Riemann tensor of R_n, $b_{ic} = b_{ci}$ is the corresponding second fundamental form and ; r denotes the covariant derivative. In the next section we will use expressions obtained in [4-8] to show that (1,2) imply the umbilical character of R_n, that is

$$b_{jc} = \frac{b}{n} g_{jc}, \quad b \equiv b^{r}{}_{r}, \quad n=3,4. \quad \textbf{(4)}$$

In fact, (4) is correct for every n, which may be seen [9] by a carefull analysis of the eigenvalue problem for b_{jr}. By substitution of (4) into (2) one may deduce:

$$R_{ijrm} = \overline{K}(g_{ir}g_{jm} - g_{im}g_{jr}), \quad \overline{K} = K + \frac{\varepsilon b^2}{n^2} \quad \textbf{(5)}$$

so, the treated R_n, n = 3,4 also turns to be a space of constant gaussian curvature $\overline{K}$.

2. UMBILICAL PROPERTY OF R_n, n = 3,4 OF EINSTEIN SPACE EMBEDDED INTO (N+ 1) -SPACE OF CONSTANT CURVATURE.

Equation (2) was studied in [5] only for the case K = O, however, we can extend the use of the same scheme to our main analysis without major difficulty; when $K \neq$ O we obtain the relation:

$$p b_{ij} = \frac{R}{6} + \frac{1}{6} R_i^m R_{mj} - \frac{1}{3} R_{imrj} R^{mr} - \frac{1}{12} R_{imrc} R_j^{mrc} + \frac{n(n-1)}{6} K R_{ij} + \\ + \frac{K}{6}\left[(n-3)R + n(n-1)(n-2)K\right] g_{ij}, \quad \textbf{(6)}$$

with

$$p = \frac{\varepsilon}{3} b^{rc} G_{rc} - \frac{\varepsilon K}{6}(n-1)(n-2) b, \quad \textbf{(7)}$$

where $R = R^j_j$ is the scalar curvature and $G_{ij} = R_{ij} - \frac{R}{2} g_{ij}$ is the Einstein tensor of R_n.

It is now convenient to split our analysis in two directions:

a). Case n=3.

Here we will see the condition that the umbilical character of R_3 implies. It is widely known [1,2] that in three dimensions the Ricci tensor determines the Riemann tensor:

$$R_{ijrc} = R_{jr} g_{ic} + R_{ic} g_{jr} - R_{ir} g_{jc} - R_{jc} g_{ir} + \frac{R}{2}(g_{ir} g_{jc} - g_{ic} g_{jr}) \quad \textbf{(8)}$$

then introduction of (1,8) into (6,7) gives (4,5) with:

$$b^2 = -9\varepsilon\left(K + \frac{R}{6}\right) > 0, \quad \overline{K} = -\frac{R}{6} \quad \text{if } \left(K + \frac{R}{6}\right) \neq 0 \quad \textbf{(9)}$$

which determines the sign of ε ***.***

b). Case n = 4.

Lanczos identities [10] reduce (6,7) to the form [4-8,11,12].

$$p b_{ij} = -\frac{1}{2} R_{ircj} G^{rc} + 2KR_{ij} + (4K^2 + \frac{KR}{6} + \frac{1}{48} K_2) g_{ij} \quad \textbf{(10)}$$

with

$$p = \frac{\varepsilon}{3} b^{ic} G_{ic} - \varepsilon b K, \qquad K_2 = {}^{*}R^{*ijrc} R_{ijrc} \quad \textbf{(11)}$$

where *R*arjc* is the double dual [2,10] of the Riemann tensor.

By substitution of (1) into (10,11) we obtain (4,5) with:

$$b^2 = -16\varepsilon(K + \frac{R}{12}) > 0, \quad \overline{K} = -\frac{R}{12}, \quad K_2 = -\frac{R^2}{6} \quad \textbf{(12)}$$

under the condition $(K + \frac{R}{12}) \neq 0$; this result (12) can be seen as a generalization of

theorem II of [13].

In this way we have showed that our relations (6,7) give a simple proof of the umbilical character of R_3 (R_4) of Einstein embedded into R_4 (R_5) of constant curvature.

REFERENCES

[1] D. Lovelock and H. Rund, *Tensors, differential forms and varíational principles,* John Wiley and Sons (1975)

[2] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact solutions of Einstein's field equations,* Cambridge University Press (1980)

[3] J. López-Bonilla, H.N. Núñez-Yepez, J.M. Rivera and A.L.Salas-Brito, Acta Phys. Slovaca 46 (1996) 87.

[4] R. Fuentes, J. López-Bonilla, G. Ovando and T. Matos, Gen. Relat. Grav. 21 (1989) 777

[5] G. González, J. López-Bonilla and M.A. Rosales, Pramana J. Phys. 42 (1994) 85.

[6] J. López-Bonilla, J. Morales and M.A. Rosales, Braz. J. Phys. 24 (1994) 522

[7] J. López-Bonilla, J.M. Rivera and H. Yee-Madeira, Braz. J. Phys. 25 (1995) 80

[8] H.N. Núñez-Yépez and J. López-Bonilla, Pramana J. Phys. 46 (1996) 219

[9] H. Rund, Ann. Mat. Pura Appl. 93 (1972) 99

[10] C. Lanczos, Ann. of Math. 39 (1938) 842

[11] J. H. Caltenco, J. López-Bonilla and G. Ovando, J. Bangladesh Acad. Sci. 25 (2001) 95.

[12] J. López-Bonilla, J. Morales and G. Ovando, Indian J. Math. 42 (2000) 309.

[13] P. Szekeres, Nuovo Cim. A43 (1966) 1062.