A Pure Mathematical Proof of the Four Color Problem

Author : **Fayez Fok Al Adeh**

President of the Syrian Cosmological Society

Tels: 00963 – 11- 2776729

00963 – 11 – 2713005

Email : hayfa @ scs – net .org

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Abstract

As stated originally the four – color problem asked whether it is always possible to color the regions of a plane map with four colors such that regions which share a common boundary (and not just a point) receive different colors. In the long and arduous history of attacks to prove the four color theorm many attempts came close, but what finally succeeded in the Apple – Haken proof of 1976 and also in the recent proof of Robertson ,Sanders , Seymour and Thomas 1997 was a combination of some old ideas and the calculating powers of modern – day computers. Thirty years after the original proof, the situation is still basically the the same,no pure mathematical proof is in sight.Now I give in my paper such a pure mathematical proof.

The four color problem asks whether it is always possible to color the regions of a plane map with four colors such that regions which share a common boundary (and not just a point) receive different colors. Coloring the regions of a plane map is really the same task as coloring the vertices of a plane graph . We place a vertex in the interior of each region (including the outer region) and connect two

such vertices belonging to neighboring regions by an edge through the common boundary.

The resulting graph G , the dual graph of the map M, is then a plane graph , and coloring the vertices of G in the usual sense is the same as coloring the regions of M. Because of this construction we may as well concentrate on vertex – coloring graphs drawn on the 2 – sphere S^2 and will do so from now on .Note that we may assume that G has no loops or multiple edges ,since these are irrelevant for coloring .

First note that adding edges can only increase the chromatic number. In other words,when H is a subgroup of G, then χ_l (H) $\leq$ χ_l (G) certainly holds . χ_l is the list chromatic number.Hence we may assume that G is connected .

Since ordinary coloring is just the special case of list coloring ,we obtain for any graph G

(1) χ **(G)** $\leq$ χ_l **(G)**

where χ **(G) is the ordinary chromatic number.**

Now we consider the given map M (including the outer region) and its' dual graph G (including the vertex in the interior of the outer region) The regions of M exhaust all of S^2 .

Thus S^2 is SO_3 – paradoxical using the regions of the map M (some of the employed elements of SO_3 may coincide with the identity of SO_3) . Now we color the vertices of the dual graph G in such a way that the two vertices at the two ends of any edge of the graph G receive different colors, and such that the number of colors used is a minimum. Let n be the minimum number of different colors used. For each color i (i = 1,2,……,n) we collect all the regions of the map

M with the property that the vertices of G in the interiors of all of these regions have the same color i . Let A_i be the union of the collection of all such regions.

(2) $A_i = \bigcup_{k=1}^{m_i} R_k$

Where R_k (K=1 ,2 ,…, m_i) are the regions of the map M with the property that the vertices of G in the interiors of all these regions have the same color i .

m_i the number of these regions R_k (K=1 ,2 ,…, m_i)

For any $f \in SO_3$ acting on S^2, and for any function f in general , we have

(3) $f(A_i) = f(\bigcup_{k=1}^{m_i} R_k) = \bigcup_{k=1}^{m_i} f(R_k)$

This means that S^2 is SO_3 – paradoxical using the new subsets A_i (i = 1,2,….,n). We call these subsets A_i (i = 1,2,….,n). the derived subsets based on minimum coloring of a given map M .

We can then obtain immediately the following easy theorm:

(4) Theorm : S^2 is SO_3 – paradoxical using the derived subsets A_i (i = 1,2,….,n). The number of these subsets n equals the minimum number of colors used .

We mention now the following known theorm:

(5) Theorm : S^2 is SO_3 – paradoxical using four regions ,and the four cannot be improved.

What about the greatest lower bound of all the possible minimum numbers of colors used in all possible cases. ?

A ccording to this last theorm (5) and in reference to theorm (4) above ,we assert that whenever an SO_3 – paradoxical decomposition of S^2 is given using derived subsets in the sense of theorm (4) above , the

greatest lower bound of all possible numbers of these subsets is four, and the four cannot be improved .

Therefore, according to theorm (4) above , the greatest lower bound of all minimum numbers of colors used to color any graph (or map) is four,and the four cannot be improved .

This concludes the proof of the four color conjecture .

References :

1 – Aigner,Martin ; Ziegler,Gunter M. (2002) Proofs from the Book, Berlin : Springer – Verlag .

2 – Wagon, Stan (1994) The Banach – Tarski Paradox , New York : Cambridge University Press.

3 – Saaty, Thomas L . ; Kainen ,Paul C . (1977) The Four Color Problem , New York : Mcgraw – Hill Book Company .